Renormalization of Four-Fermion Operators in Asymptotically Free Theories*

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Assuming that the strong interactions are described by $SU(N)$ color-gauge theories, one can calculate the logarithmic "enhancement" factors of various operators which contribute to the lowest order weak and e. m. transitions between hadronic states. In this paper we compute the enhancement factors of four-fermion operators and show that the usually assumed V-A type four-fermion operators do not have larger enhancement factors than some of the other types, such as the scalar, pseudoscalar or tensor types.

I. INTRODUCTION

BY now it is well known that non-Abelian gauge theories are asymptotically free[1]. In such theories, therefore, perturbation calculations become useful in the asymptotic region. For example, when such theories are adopted for strong interactions, and are combined with suitable unified gauge models of weak and e. m. interactions, the structure of lowest order weak and e. m. transitions between hadronic states can be rigorously analyzed by using the renormalization group technique and the perturbation method just mentioned[2]. In particular, Weinberg[3] has studied, in a class of models to be described in Sec. II, the structure of the transition amplitudes of order α, and shown that they merely consist of the ordinary one-photon-exchange part and certain weak corrections to the quark mass matrix. This result not only explains the observed pattern of the parity and strangeness conserving but isospin breaking phenomena in the hadronic world, but also offers a plausible ground for attacking the longstanding problems of the proton-neutron mass difference and of the $\eta\rightarrow3\pi$ process.[4]

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(1) D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973), H. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

(2) H. Georgi and H. Politzer, Phys. Rev. D9, 416 (1974), D. Gross and F. Wilczek, Phys. Rev. D9, 980 (1974), see also Refs. 3 and 5.

(3) S. Weinberg, Phys. Rev. D8, 4462 (1973).

(4) S. Weinberg, Rev. Mod. Phys. 46, 255 (1974).

Similarly, weak and e. m. transition amplitudes of order $G_F \sim \alpha/M_W^2$ have also been analyzed in these models (G_F denotes the Fermi coupling constant and M_W the mass of the charged W-boson.) Under the additional assumption that the Higgs scalar mass is much greater than the mass scale of the quarks m, it has been shown[5] that the strangeness changing amplitudes get contributions mainly from the W-boson exchange processes, and that they can be written as

$$(i \to f)_{\Delta S \neq 0} = \frac{\alpha}{M_W^2} \sum_n C_n \left(\ln \frac{M_W^2}{m^2} \right)^{d_n} \langle f | O_n | i \rangle$$

$$+ \text{higher order terms in } (m^2/M_W^2), \qquad (1)$$

where each O_n is a local operator of dimension 5 or 6 that appears in the short distance expansion of the time-ordered product of two hadronic weak currents, d_n is a characteristic dimension of the operator O_n, to be defined in Sec. II, and C_n is a coefficient which unfortunately cannot be determined perturbatively.

Equation (1) shows that the strangeness changing amplitudes are dominated by those operators with particularly large d_n, assuming that the products of the coefficients C_n and the matrix elements $\langle f | O_n | i \rangle$ are more or less of the same order of magnitude[6]. This gives rise to a possible mechanism for enhancing the contributions of certain operators with definite quantum numbers, e. g., $\Delta I = 1/2$ versus $\Delta I = 3/2$. This possible mechanism for the $\Delta I = 1/2$ enhancement in nonleptonic weak interactions has been studied by Gaillard and Lee[7], and by Altarelli and Maiani[8]. They found that in a general class of models enhancement of the $\Delta I = 1/2$ part does occur, and typical values of the enhancement factor $\left(\ln \frac{M_W^2}{m^2} \right)^{d_{1/2} - d_{3/2}}$ are 5-10. However, it is not conclusive that this enhancement mechanism is enough to account for the experimentally obseverd $\Delta I = 1/2$ pattern, because actually one does not know much about the coefficients C_n that appear in Eq. (1), which may well upset the enhancement factor.

In the analysis of Refs. (7) and (8), the relevant operators O_n in Eq. (1) are assumed to be the four-fermion operators that have the Lorentz structure of the product of two *V-A* or *V+A* type currents. The restriction to operators of this type is presumably based on certain notions of universality of the weak interactions in the old current-current approach. However, there is no a priori reason to exclude from the set $\{O_n\}$ operators with different Lorentz structure, such as products of two scalar, or pseudo-scalar. or tensor currents[9]. Thus both (V, A) type and $(S, P, T,)$ type operators can in principle appear in the expansion, Eq. (1). The aim of this paper is to investigate whether the (V, A) type four-fermion operators are more important than the (S, P, T) type in the expansion, in the sense that the former have bigger characteristic dimensions d_n than the latter. This is essentially the same type of analysis carried out in Ref. 8 for studying the $\Delta I = 1/2$ enhancement factor. The only difference is that

(5) V. Mathur and H. Yen, Phys. Rev. **D8**, 3569 (1973).

(6) This assumption would be valid if the strong interactions were turned off.

(7) M. Gaillard and B. Lee, Phys. Rev. Lett. 33, 108 (1974).

(8) G. Altarelli and L. Maiani, Phys. Lett. **B52**, 351 (1974).

(9) See a related discussion on neutral currents by B. Kayser et al., Phys. Lett. **B52**, 385 (1974).

in this paper we are comparing the characteristic dimensions of the (V, A) type operators versus the (S, P, T) type operators, regardless of their isospin or hadronic $SU(3)$ properties. In Sec. II, we will define the framework of our discussion and then define the characteristic dimension of an operator and describe the procedure for its calculation. The detailed techniques for this calculation will be presented in the Appendices. Sec. III contains the final results and a discussion on their relation to the Fierz transformations.

II. CHARACTERISTIC DIMENSIONS OF FOUR-FERMION OPERATORS

First we describe the class of models that has been discussed by Weinberg[3], which is also to be used in our discussion. These are quark models of strong, weak and e. m. interactions with the following specifications:

(1) all interactions are mediated by gauge fields, corresponding to a total gauge group $G_S \otimes G_W$;

(2) the strong interaction gauge symmetry G_S is realized by a set of neutral, non-Abelian massless gauge fields (the color gluons). Gs is an exact symmetry, and will be specified to be $SU(N)$;

(3) the weak and e. m. interactions are described by any reasonable unified gauge models, such as the Weinberg-Salam type models[10]. G_W is spontaneously broken, and is usually taken to be $SU(2)\times U(1)$;

(4) the number of quark species is assumed to be n. All quarks but p, n, λ are SU(3) singlets, carrying various charm-like_ quantum numbers.

In this class of models, the Lagrangian for strong interactions has the following general form:

$$L_S=-\sum_j \bar{\psi}_j \gamma_\mu(\partial_\mu+igT_A G_{A\mu})\psi_j-\sum_j \bar{\psi}_j m_j \psi_j-\frac{1}{4}G_{A\mu\nu}G_{A\mu\nu},$$

where each ψ_j is an N-component column vector in the $SU(N)$ color space and corresponds to the j^{th} type quark $(j=1, 2, \ldots n)$, T_A are representations of the color $SU(N)$ group, $(A=1,2,\ldots,N^2-1)$, and $G_{A\mu}$ are the color gauge fields. The Lagrangian L_S is invariant under the $SU(N)$ color gauge transformations. Some useful properties of the $SU(N)$ matrices T_A are collected in Appendix II.

The strong interaction theory described by L_S is asymptotically free. In such a theory, it can be shown[11] that the asymptotic behavior of a general renormalized Green ' s function whose external momenta are scaled to infinity simultaneously, is

$$\Gamma_R(Kp_0, g_R, m_R, \mu) \underset{K\to\infty}{\sim} K^{D_\Gamma}(\ln K)^{-L_\Gamma} * [\text{power series in } (\ln K)^{-1}]$$

provided[12] $\Gamma_R(p_0, g_R, 0, \mu)\neq 0$. In this formula, D_Γ is the naive dimension of Γ_R, and $L_\Gamma=C_\Gamma/2b$, where $2b=\frac{1}{8\pi^2}\left[\frac{11N-2n}{3}\right]>0$ is fixed for a given gauge

(10) See, for example, J. Bernstein, Rev. Mod. Phys. 46, 7 (1974).

(11) S. Weinberg, Phys. Rev. **D8, 3497 (1973).**

(12) If this condition is not satisfied, the asymptotic expression for Γ_R gets modified, but its essential features remain unchanged. There would be no problem in handling this minor complication. See Ref. 11 for details.

group $SU(N)$ and a given number n of quark species[13], and C_Γ is determined from the renormalization constant of Γ_R in the following way[11]. After renormalizing the coupling constant and the Green's function with cutoff Λ and subtraction point scale μ,

$$g_R = g_R(g, \Lambda/\mu),$$

$$\Gamma_R(p, g_R, m_R, \mu) = Z_\Gamma(g, \Lambda/\mu)\cdot\Gamma(p, g, m, \Lambda),$$

one calculates the quantity γ_Γ defined by

$$\gamma_\Gamma(g_R) \equiv \mu\frac{\partial}{\partial\mu}\ln Z_\Gamma(g, \Lambda/\mu)|_{g \text{ and } \Lambda \text{ fixed}},$$

and expands it into power series of the renormalized coupling constant g_R,

$$\gamma_\Gamma(g_R) = C_\Gamma g_R^2 + O(g_R^4).$$

The coefficient of the g_R^2 term in the series is what we need for the determination of the asymptotic behavior of Γ_R.

The same analysis can be extended to the case in which only a subset of the external momenta of a Green's function are made infinitely large'"). Suppose Γ_{AB} is the Green's function under consideration, where A denotes the subset of the external legs whose momenta k are made infinitely large, and B denotes the remaining legs whose momenta q are kept fixed. Then Wilson's expansion leads to the following approximation :

$$\Gamma_{AB,R}(k, q, g_R, m_R, \mu) \underset{k\to\infty}{\sim} \sum_O U_{AO}(k, g_R, m_R, \mu)\Gamma_{BO,R}(q, g_R, m_R, \mu),$$

with finite C-number coefficient functions U_{AO}.Γ_{BO} denotes the Green's function for the external legs B, with an extra zero-momentum 0 vertex. Using renormalization group equation technique, one can show that

$$U_{AO}(Kk_0, g_R, m_R, \mu) \underset{K\to\infty}{\sim} K^{D_{AO}}(\ln K)^{L_{AO}}\cdot[\text{power series in } (\ln K)^{-1}],$$

provided $U_{AO}(k_0, g_R, 0, \mu) \neq 0$, where D_{AO} is the naive dimension of U_{AO}, and $L_{AO} = C_{AO}/2b$, with C_{AO} defined by the following self-explanatory set of equations :

$$\Gamma_{AB,R}(k, q, g_R, m_R, \mu) = Z_{AB}(g, \Lambda/\mu)\cdot\Gamma_{AB}(k, q, g, m, \Lambda),$$

$$\Gamma_{BO,R}(q, g_R, m_R, \mu) = Z_{BO}(g, \Lambda/\mu)\cdot\Gamma_{BO}(q, g, m, \Lambda),$$

$$\gamma_{AO} \equiv \mu\frac{\partial}{\partial\mu}\ln[Z_{BO}/Z_{AB}]|_{g \text{ and } \Lambda \text{ fixed}},$$

$$\gamma_{AO}(g_R) = C_{AO} g_R^2 + O(g_R^4).$$

Sometimes γ_{AO} becomes independent of the set A, which is the case when, for example, the set A consists only of conserved or partially conserved vector

(13) D. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973). Notice that the matrix T_A defined in their paper is different from our T_A by a factor of two.

or axial-vector currents. Then the number L_{AO} depends solely on the operator 0. This situation will occur in our following calculation.

Now let us combine the Lagrangian L_S with a unified gauge model of weak and e. m. interactions,

$$L_{wk-em}=J_{\alpha\mu}W_{\alpha\mu}+S_i\phi_i+\cdots,$$

with $J_{\alpha\mu}$ and S_i the hadronic weak-e. m. currents. It has been shown[5] that of all possible processes inducing the lowest order weak and e. m. transitions, the exchanges of one weak gauge boson are the most interesting. The amplitudes for these exchanges are

$$T_W\sim\int d^4kF_{\alpha\mu,\ \beta\nu}(k)\cdot\Delta_{\alpha\mu,\beta\nu}(k),$$

where

$$F_{\alpha\mu,\ \beta\nu}(k)=\int d^4x\langle f|T\{J_{\alpha\mu}(x)J_{\beta\nu}(o)\}|i\rangle e^{ikx},$$

and $\Delta_{\alpha\mu,\ \beta\nu}(k)$ are the propagators of weak gauge bosons W^{μ}_{α}. When Wilson's expansion for the product of currents $T\{J_{\alpha\mu}(x)J_{\beta\nu}(o)\}$ is used, the matrix element T_W can be shown to have the following structure[5,8]:

$$T_W=\frac{\alpha}{M_W^2}\sum_n C_n\left[\ln\left(\frac{M_W^2}{m^2}\right)\right]^{d_n}\cdot\langle f|O_n|i\rangle$$
$$+\text{ higher order terms in } m^2/M_W^2,$$

where the summation is over the complete set of local operators that should appear in the expansion of $T\{J_{\alpha\mu}(x)J_{\beta\nu}(o)\}$. Each of the operators 0, consists of a certain product of the field operators $\psi,\bar{\psi},G_{A\mu}$ and their derivatives at a single space-time point.

For strangeness changing transitions to order G_F, the relevant operators O_n are the 4-fermion operators which are color singlet and which are Lorentz scalar or pseudoscalar. There are 20 such operators in total:

$$O_1=\bar{\psi}_1\gamma_\mu\psi_2\bar{\psi}_3\gamma_\mu\psi_4 \qquad O_2=\bar{\psi}_1i\gamma_\mu\gamma_5\psi_2\bar{\psi}_3i\gamma_\mu\gamma_5\psi_4$$

$$O_3=\bar{\psi}_1\gamma_\mu\psi_2\bar{\psi}_3i\gamma_\mu\gamma_5\psi_4 \qquad O_4=\bar{\psi}_1i\gamma_\mu\gamma_5\psi_2\bar{\psi}_3\gamma_\mu\psi_4$$

$$O_5=\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4 \qquad O_6=\bar{\psi}_1\gamma_5\psi_2\bar{\psi}_3\gamma_5\psi_4$$

$$O_7=\bar{\psi}_1\sigma_{\mu\nu}\psi_2\bar{\psi}_3\sigma_{\mu\nu}\psi_4 \qquad O_8=\bar{\psi}_1\psi_2\bar{\psi}_3\gamma_5\psi_4$$

$$O_9=\bar{\psi}_1\gamma_5\psi_2\bar{\psi}_3\psi_4 \qquad O_{10}=-\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\psi}_1\sigma_{\mu\nu}\psi_2\bar{\psi}_3\sigma_{\alpha\beta}\psi_4$$

$$Q_1=\bar{\psi}_1\gamma_\mu T_A\psi_2\bar{\psi}_3\gamma_\mu T_A\psi_4 \qquad Q_2=\bar{\psi}_1i\gamma_\mu\gamma_5T_A\psi_2\bar{\psi}_3i\gamma_\mu\gamma_5T_A\psi_4$$

$$Q_3=\bar{\psi}_1\gamma_\mu T_A\psi_2\bar{\psi}_3i\gamma_\mu\gamma_5T_A\psi_4 \qquad Q_4=\bar{\psi}_1i\gamma_\mu\gamma_5T_A\psi_2\bar{\psi}_3\gamma_\mu T_A\psi_4$$

$$Q_5=\bar{\psi}_1T_A\psi_2\bar{\psi}_3T_A\psi_4 \qquad Q_6=\bar{\psi}_1\gamma_5T_A\psi_2\bar{\psi}_3\gamma_5T_A\psi_4$$

$$Q_7=\bar{\psi}_1\sigma_{\mu\nu}T_A\psi_2\bar{\psi}_3\sigma_{\mu\nu}T_A\psi_4 \qquad Q_8=\bar{\psi}_1T_A\psi_2\bar{\psi}_3\gamma_5T_A\psi_4$$

$$Q_9=\bar{\psi}_1\gamma_5T_A\psi_2\bar{\psi}_3T_A\psi_4 \qquad Q_{10}=-\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\psi}_1\sigma_{\mu\nu}T_A\psi_2\bar{\psi}_3\sigma_{\alpha\beta}T_A\psi_4.$$

In each of these operators, the four fermion fields $\psi_1, \psi_2, \psi_3, \psi_4$, are generally different from each other. The main purpose of this paper, is to find the characteristic dimensions of these *20* operators. In order to do this, one has to compute the matrix elements $\langle p_1, p_3$ IO, $|p_2, p_4\rangle$ to second order in g (See Fig. 1). The four external momenta $1, p_2, p_3, p_4 are to be taken as space-like. We can choose $p_1^2=p_2^2=p_3^2=p_4^2=\mu^2$ where μ is the subtraction point scale. We shall carry out the calculation in the Landau gauge of the gluon fields, so that the renormalization constant for each quark field ψ becomes $Z_\psi=1+O(g^4)$. That is, $Z_\psi=1$ to order g^2, and hence we need not worry about the renormalization of the four externai legs in the above graphs" '). We thus have the simple relation

$$\langle p_1, p_3 | O|p_2, p_4\rangle=Z_0^{-1}\langle p_1, p_3 | O_R |p_2, p_4\rangle,$$

where Z_0 is the renormalization constant of the operator 0.

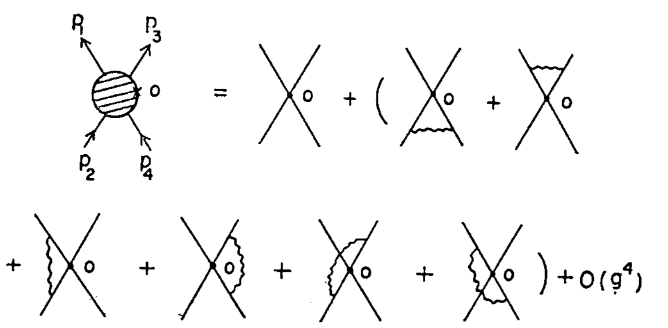

Fig. 1. Feynman graphs for the unrenormalized matrix element of 0 up to the second order of g.

However, there is a subtlety in the renormalization procedure of a general operator 0. When one calculates the matrix element $\langle p_1, p_3|O|p_2, p_4\rangle$ to higher orders than the lowest, in general it will not come out in a form similar to the lowest-order matrix element. That is, the renormalization constant does not factor out automatically in the higher order matrix elements. In this case we say that the operator 0 is not multiplicatively renormalized. It turns out that usually several operators get mixed up when renormalized, and only certain specific linear combinations of them are separately multiplicatively renormalized[14]. This complication occurs for our operators $\{O_i\}$ and $\{Q_i\}$. Therefore our task is to find the correct combinations of the O's and the Q's, and for each combination to calculate its corresponding renormalization constant. From this we can obtain the characteristic dimension d for this particular combination of the operators. The details of the calculation will be left for Appendix III, and the final results are presented in the following section.

(14) D. Gross and F. Wilczek, Phys. Rev. **D9, 980** (1974).

III. RESULTS

The results of the calculation are listed in Table I. There must be 20 combinations of the 20 operators $O_1 \ldots Q_{10}$ that are multiplicatively renormalized. Here we list in the table only ten of them, together with their corresponding dimensions d, and the value of the d's for the special case of $N=3$ and $n=4$. The missing ten combinations can be recovered from those in the table by the following simple substitutions:

$$O_1 \to O_3,\ O_2 \to -O_4,\ O_{5,6,7} \to O_{8,9,10}$$

$$Q_1 \to Q_3,\ Q_2 \to -Q_4,\ Q_{5,6,7} \to Q_{8,9,10}.$$

The characteristic dimension d remains unchanged for each substitution. For example, the characteristic dimension for the combination $(O_3 - O_4) + \frac{N}{2}(Q_3 - Q_4)$ is identical to that for the combination $(O_1 + O_2) + \frac{N}{2}(Q_1 + Q_2)$, both being given by $\frac{1}{2b}\left(\frac{3}{8\pi^2}\right)\frac{N^2-1}{N}$.

Table I. Multiplicatively renormalizable combinations of four-fermion operators and their characteristic dimensions

Combination of operators	Characteristic dimension d	Value of d for $N=3, n=4$
$(O_1+O_2)+\frac{N}{2}(Q_1+Q_2)$	$B(N^2-1)/N$	0.96
$(O_1+O_2)-\frac{N}{2(N^2-1)}(Q_1+Q_2)$	$-B/N$	−0.12
$(O_1-O_2)-\frac{N}{2(N-1)}(Q_1-Q_2)$	$B(N+1)/N$	0.46
$(O_1-O_2)+\frac{N}{2(N+1)}(Q_1-Q_2)$	$-B(N-1)/N$	-0.24
(O_5-O_6)	$B(N^2-1)/N$	0.96
(Q_5-Q_6)	$-B/N$	-0.12
$\alpha_i(O_5+O_6)+$	$B(N^2-1+3N+R)/3N$	1.30
$\beta_i(Q_5+Q_6)+$	$B(N^2-1-3N+R)/3N$	0.58
$\gamma_i O_7+\delta_i Q_7$	$B(N^2-1+3N-R)/3N$	0.06
$(i=1,2,3,4)$	$B(N^2-1-3N-R)/3N$	-0.66

$B \equiv \frac{1}{2b}\left(\frac{3}{8\pi^2}\right) = \frac{9}{11N-2n}$, $R \equiv (4N^4-11N^2+16)^{1/2}$. The expressions for α_i, β_i, γ_i and δ_i can be found in Appendix III.

In the first four lines of Table I we recover the results first obtained by Altarelli and Maiani[8]. Recall that $O_1, \ldots O_4$ and $Q_1, \ldots Q_4$ involve products of vector or axial-vector currents, while $O_5, \ldots O_{10}$ and $Q_5, \ldots Q_{10}$ involve products of scalar, pseudoscalar or tensor currents. For the special case $N=3$ and $n=4$, which is the case most often studied, we see from the numerical values of d in the table that one particular combination of (S, P, T) type operators has the largest characteristic dimension 1.3, while another combination of (S, P, T) typ operators has the smallest, -0.66. Thus the *(V, A)* type operators do. not dominate the (S, P, T) type operators by having a larger characteristic dimension. This last statement remains true if we vary the color gauge group $SU(N)$ through all integers $N \geqslant 2$, as can be easily checked using the expressions of the characteristic dimensions in Table I.

An interesting observation is that several different combinations of the four-fermion operators may have the same characteristic dimensions. We see from Table I that a pair of combinations, $(O_5 - O_6)$ and $(O_1 + O_2) + \frac{N}{2}(Q_1 + Q_2)$, correspond to the same characteristic dimension, and so do the pair $(Q_5 - Q_6)$ and $(O_1 + O_2) \frac{N}{2(N^2-1)}(Q_1 + Q_2)$. This is not a mere coincidence but a consequence of a -generalized Fierz reordering theorem, of which we present a detailed analysis in Appendix IV. We see, from the analysis therein that, for example, the operators $2N(O_5 - O_6)$ and $(O_1 + O_2) + \frac{N}{2}(Q_1 + Q_2)$ transform into each other under the Fierz rearrangement, and hence must have the same characteristic dimension in order to be consistent with the results of our calculations. In other words, the characteristic dimension of a four-fermion operator must be invariant under the Fierz transformations. This has been checked for all the operators in Table I, as is done in Appendix IV.

APPENDIX I

In this appendix we first list the convention of the T-matrices used in this paper to avoid any unnecessary confusion. Then we explain some notations to be used later, and finally we collect some useful reduction formulae of the γ-matrices.

For the Minkowski metric we use $g_{\mu\nu} = 1, 1, 1, -1$, and for the T-matrices,

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\nu\mu}, \ \vec{\gamma} = -i\beta\vec{\alpha}, \gamma_4 = \beta, \ \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4,$$

$$\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu \ \gamma_\nu], \ \gamma_\mu^+ = \gamma_\mu, \ \gamma_5^+ = \gamma_5, \ \gamma_\mu^2 = \gamma_5^2 = 1.$$

The Feynman rules for the propagators and the vertex in our calculation are as follows.

$$\text{fermion propagator} \ \text{———} = \frac{-i}{i\gamma \cdot p + m},$$

$$\text{gluon propagator} \ \underset{\mu \quad\quad \nu}{\overset{A \quad\quad B}{\sim\sim\sim\sim}} = \frac{-i}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\delta_{AB},$$

$$\text{vertex} \ \frac{\wr}{A\mu} = g\gamma_\mu T_A.$$

Let S, P, T, *V*, *A* symbolically denote, respectively, the matrices 1, $\gamma_5, \sigma_{\mu\nu}$,

$\gamma_\mu, i\gamma_\mu\gamma_5$ sandwiched between two general quark bispinors. For example, the lowest order matrix element $\langle p_1, p_3|O_5|p_2, p_4\rangle = \langle O_5\rangle = \bar{u}(p_1)u(p_2)\bar{u}(p_3)u(p_4)$, is a product of two scalars and hence will be abbreviated as SS. Similarly, the lowest order matrix element $\langle p_1, p_3|Q_5|p_2, p_4\rangle = \langle Q_5\rangle = \bar{u}(p_1)T_A u(p_2)\cdot\bar{u}(p_3)T_A u(p_4)$ will be abbreviated as S'S'. Both O_5 and Q_5 are color singlets, but the singlet O_5 is a result of the product of two color singlets, $1\times1=1$, while Q_5 is the singlet that is contained in the product of two octets, $8\times8=1+8+8+10+\overline{10}+27$, if $SU(3)$ is assumed for the color gauge group. Hence we will use S to denote the color singlet, Lorentz scalar current, and S' the color octet, Lorentz scalar current. (when $N=3$). For other currents. similar notations are used:

$\langle O_1\rangle = VV$	$\langle O_2\rangle = AA$	$\langle O_3\rangle = VA$	$\langle O_4\rangle = AV$
$\langle O_5\rangle = SS$	$\langle O_6\rangle = PP$	$\langle O_7\rangle = TT$	
$\langle O_8\rangle = SP$	$\langle O_9\rangle = PS$	$\langle O_{10}\rangle = ET$	
$\langle Q_1\rangle = V'V$	$\langle Q_2\rangle = A'A'$	$\langle Q_3\rangle = V'A'$	$\langle Q_4\rangle = A'V'$
$\langle Q_5\rangle = S'S$	$\langle Q_6\rangle = P'P'$	$\langle Q_7\rangle = T'T'$	
$\langle Q_8\rangle = S'P'$	$\langle Q_9\rangle = P'S'$	$\langle Q_{10}\rangle = E'T'$,	

where

$$T = \bar{u}_1\sigma_{\mu\nu}u_2,\quad E = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{u}_1\sigma_{\alpha\beta}u_2.$$

The following is a list of easily proven but useful formulae for reducing products of r-matrices into S, P, T, V, A.

(a) reduction' of the product $\gamma_\mu\gamma_\alpha\Gamma\gamma_\alpha\gamma_\mu$:

$$\gamma_\mu\gamma_\alpha\gamma_\alpha\gamma_\mu = 16,\qquad \gamma_\mu\gamma_\alpha\gamma_5\gamma_\alpha\gamma_\mu = 16\,\gamma_5,$$

$$\gamma_\mu\gamma_\alpha(\gamma_\nu)\gamma_\alpha\gamma_\mu = 4\,\gamma_\nu,\qquad \gamma_\mu\gamma_\alpha(i\gamma_\nu\gamma_5)\gamma_\alpha\gamma_\mu = 4(i\gamma_\nu\gamma_5),$$

$$\gamma_\mu\gamma_\alpha(\sigma_{\nu\beta})\gamma_\alpha\gamma_\mu = 0.$$

(b) reduction of the product $\Gamma\gamma_\alpha\gamma_\mu$:

$$\gamma_\alpha\gamma_\mu = \delta_{\alpha\mu} + i\sigma_{\alpha\mu},\quad \gamma_5\gamma_\alpha\gamma_\mu = \delta_{\alpha\mu}\gamma_5 - \frac{i}{2}\epsilon_{\alpha\mu\nu\beta}\sigma_{\nu\beta},$$

$$\gamma_\nu\gamma_\alpha\gamma_\mu = \delta_{\alpha\mu}\gamma_\nu + \delta_{\nu\alpha}\gamma_\mu - \delta_{\nu\mu}\gamma_\alpha + i\epsilon_{\nu\alpha\mu\beta}(i\gamma_\beta\gamma_5),$$

$$(i\gamma_\nu\gamma_5)\gamma_\alpha\gamma_\mu = \delta_{\alpha\mu}(i\gamma_\nu\gamma_5) + \delta_{\nu\alpha}(i\gamma_\mu\gamma_5) - \delta_{\nu\mu}(i\gamma_\alpha\gamma_5) - i\epsilon_{\nu\alpha\mu\beta}\gamma_\beta,$$

$$(\sigma_{\nu\beta})\gamma_\alpha\gamma_\mu = i\delta_{\nu\alpha}\delta_{\beta\mu} - i\delta_{\nu\mu}\delta_{\beta\alpha} - i\epsilon_{\nu\beta\alpha\mu}\gamma_5 + \sigma_{\nu\beta}\delta_{\alpha\mu} - \delta_{\nu\alpha}\sigma_{\beta\mu} + \delta_{\nu\mu}\sigma_{\beta\alpha} + \delta_{\beta\alpha}\sigma_{\nu\mu} - \delta_{\beta\mu}\sigma_{\nu\alpha}.$$

(c) also useful is the relation

$$\gamma_\mu\gamma_\alpha\Gamma = (\Gamma\gamma_\alpha\gamma_\mu)^+ \text{ for } \Gamma = 1,\ \gamma_5,\ \gamma_\nu,\ i\gamma_\nu\gamma_5,\ \sigma_{\nu\beta}.$$

(d) $\epsilon_{1234}=1,$

$\epsilon_{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\sigma\tau}=2(\delta_{\mu\sigma}\delta_{\nu\tau}-\delta_{\mu\tau}\delta_{\nu\sigma}),$

$\epsilon_{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\mu\tau}=6\delta_{\nu\tau},$

$\epsilon_{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\mu\nu}=24.$

APPENDIX II

This appendix contains some useful relations about the $SU(N)$ matrices T_A.

Let $\{T_A\}$ be the $SU(N)$ matrices, $(A=1,2,\cdots,N^2-1)$, forming the $SU(N)$ algebra: $[T_A,T_B]=iC_{ABC}T_C$. They satisfy the following properties :

$$\mathrm{tr}(T_A)=0,\ \mathrm{tr}(T_AT_B)=T(R)\delta_{AB},\ \sum_A T_A^2=C_2(R)1,$$

and

$$\sum_{A,B} C_{ABC}C_{ABD}=C_2(G)\delta_{CD},$$

where 1 is the $N\times N$ unit matrix, and T(R), $C_2(R)$ and $C_2(G)$ are characteristics of the representation. We shall choose the normalization of T_A such that T(R) =2. Then $C_2(G)=4N$.

Since any $N\times N$ matrix can be expressed as a linear combination of 1 and T_A, we can write

$$\{T_A,T_B\}=e_{AB}1+D_{ABC}T_C,$$

with undetermined coefficients e_{AB} and D_{ABC}. It is easy to show that $e_{AB}=\frac{4}{N}\delta_{AB}$, and $D_{ABC}=\frac{1}{2}tr(\{T_A,T_B\}T_C)$. Thus D_{ABC} is totally symmetric in the three indices A, B, C. From $\{T_A,T_A\}=e_{AA}1+D_{AAC}T_C$, it follows that $C_2(R)=\frac{2}{N}(N^2-1)$ and $\sum_A D_{AAC}=0$. From $T_AT_B=\frac{2}{N}\delta_{AB}1+\frac{1}{2}D_{ABC}T_C+\frac{i}{2}C_{ABC}T_C$, one can calculate $\sum_B T_AT_B^2$ in two different ways:

$$\sum_B(T_AT_B)T_B=\frac{2}{N}T_A+\frac{1}{4}\sum_{B,C}(D_{ABC}D_{DBC}+C_{ABC}C_{DBC})T_D,$$

or,

$$T_A(\sum_B T_B^2)=\frac{2}{N}(N^2-1)T_A.\quad \text{Thus we find}$$

$$\sum_{B,C}D_{ABC}D_{DBC}=\frac{4}{N}(N^2-4)\delta_{AD}.$$

From these properties of T_A, we deduce the following useful relations:

$$\sum_A(T_AT_BT_A)=-\frac{2}{N}T_B,\qquad \sum_A T_A^2=\frac{2}{N}(N^2-1),$$

$$\sum_{A,B}(\bar\psi_1 T_A T_B\psi_2)(\bar\psi_3 T_A T_B\psi_4) = -\frac{4}{N}\sum_A(\bar\psi_1 T_A\psi_2)(\bar\psi_3 T_A\psi_4)$$

$$+\frac{4}{N^2}(N^2-1)(\bar\psi_1\psi_2)(\bar\psi_3\psi_4),$$

$$\sum_{A,B}(\bar\psi_1 T_A T_B\psi_2)(\bar\psi_3 T_B T_A\psi_4) = \frac{2}{N}(N^2-2)\sum_A(\bar\psi_1 T_A\psi_2)(\bar\psi_3 T_A\psi_4)$$

$$+\frac{4}{N^2}(N^2-1)(\bar\psi_1\psi_2)(\bar\psi_3\psi_4).$$

In the text and in the following appendices, we will often omit the summation symbol $\sum_A$ when the index A appears twice in a product.

APPENDIX III

In this appendix we will take one particular operator $Q_1 \equiv \bar\psi_1\gamma_\mu T_A\psi_2\bar\psi_3\gamma_\mu T_A\psi_4$ and compute *one* of the second order graphs for the matrix element $\langle p_1, p_3 | Q_1 | p_2, p_4\rangle$, just to illustrate how the general calculation goes. We first write down the amplitude for the graph (Fig. 2), using the Feynman rules of Appendix I,

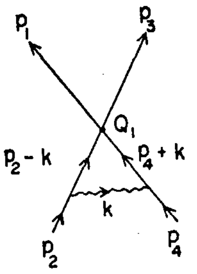

Fig. 2. A particular second order Feynman graph.

$$I_1 = \int\frac{d^4k}{(2\pi)^4}\left[\bar u_1\gamma_\mu T_A\cdot\frac{-i}{i\gamma\cdot(p_2-k)}\cdot g\gamma_\alpha T_B u_2\right]$$

$$\cdot\left[\bar u_3\gamma_\mu T_A\frac{-i}{i\gamma\cdot(p_4+k)}\cdot g\gamma_\beta T_B u_4\right]\cdot\frac{-i}{k^2}\left[\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right]$$

$$= -ig^2\int\frac{d^4k}{(2\pi)^4}\left[\bar u_1\gamma_\mu T_A\cdot\frac{\gamma\cdot(p_2-k)}{(p_2-k)^2}\cdot\gamma_\alpha T_B u_2\right]$$

$$\cdot\left[\bar u_3\gamma_\mu T_A\cdot\frac{\gamma\cdot(p_4+k)}{(p_4+k)^2}\cdot\gamma_\beta T_B u_4\right]\cdot\frac{1}{k^2}\left[\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right].$$

The masses of the fermions have been set to zero, because we are using the zero-mass renormalization procedure' "). Since we are in fact only interested in the quantity $\gamma \equiv \mu\frac{\partial}{\partial\mu}\ln Z(g, \Lambda/\mu)|_{g,\Lambda}$ fixed, which is also equal to $-\Lambda\frac{\partial}{\partial\Lambda}\ln Z(g,\Lambda/\mu)|_{g,\mu}$ fixed, all we need to calculate is the A-dependent part of the renormalization constant Z. That is, we want to obtain the coefficient C in the expression

$$Z(g,\Lambda/\mu)=1+g^2(C\ln\Lambda/\mu+\text{finite part})+O(g^4).$$

Therefore, to extract the logarithmic divergence of the integral I_1, we can replace each of the factors $(p_2-k)^2$ and $(p_4+k)^2$ in the integrand by k^2, and retain only the -divergent terms in the integrand:

$$I_1\rightarrow ig^2\int\frac{d^4k}{(2\pi)^4}[\bar{u}_1\gamma_\mu T_A(\gamma\cdot k)\gamma_\alpha T_B u_2]\cdot$$
$$\cdot[\bar{u}_3\gamma_\mu T_A(\gamma\cdot k)\gamma_\beta T_\beta u_4]\cdot\frac{1}{(k^2)^3}\left[\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right].$$

After contracting the indices α,β, we can replace the tensor $k_\nu k_\sigma$ in the integrand by $\frac{1}{4}k^2\delta_{\nu\sigma}$ according to the usual rules, and get

$$I_1=ig^2\int\frac{d^4k}{(2\pi)^4}\Big\{\frac{1}{4}[\bar{u}_1 T_A T_B\gamma_\mu\gamma_\nu\gamma_\alpha u_2]\cdot[\bar{u}_3 T_A T_B\gamma_\mu\gamma_\nu\gamma_\alpha u_4]$$
$$-[\bar{u}_1 T_A T_B\gamma_\mu u_2][\bar{u}_3 T_A T_B\gamma_\mu u_4]\Big\}\frac{1}{(k^2)^2}.$$

Now we make a Wick rotation in the k_4 plane and integrate over the solid angle, with $\int d\Omega=2\pi^2$ in the four dimensional Euclidean space. Using then the cut-off $\int^\Lambda\frac{k^3dk}{k^4}=\ln\Lambda$, we get

$$I_1=-g^2\frac{\ln\Lambda}{8\pi^2}\Big\{\frac{1}{4}[\bar{u}_1 T_A T_B\gamma_\mu\gamma_\nu\gamma_\alpha u_2][\bar{u}_3 T_A T_B\gamma_\mu\gamma_\nu\gamma_\alpha u_4]$$
$$-[\bar{u}_1 T_A T_B\gamma_\mu u_2][\bar{u}_3 T_A T_B\gamma_\mu u_4]\Big\}.$$

Finally, using the reduction formulae of Appendix I and II, we can reduce I_1 to

$$I_1=g^2\frac{\ln\Lambda}{8\pi^2}\Big\{\frac{6}{N}[\bar{u}_1\gamma_\mu T_A u_2\bar{u}_3\gamma_\mu T_A u_4$$
$$-\bar{u}_1 i\gamma_\mu\gamma_5 T_A u_2\bar{u}_3 i\gamma_\mu\gamma_5 T_A u_4]$$
$$-\frac{6(N^2-1)}{N^2}[\bar{u}_1\gamma_\mu u_2\bar{u}_3\gamma_\mu u_4-\bar{u}_1 i\gamma_\mu\gamma_5 u_2\bar{u}_3 i\gamma_\mu\gamma_5 u_4]\Big\}.$$

This expression will be abbreviated, according to the notations explained in Appendix I, as

$$I_1=g^2\frac{\ln\Lambda}{8\pi^2}\Big\{\frac{6}{N}[V'V'-A'A']-\frac{6(N^2-1)}{N^2}[VV-AA]\Big\}.$$

This is the result corresponding to the single graph of Fig. 2. When all second order graphs are calculated and summed up, we get

$$\langle p_1, p_3 | Q_1 | p_2, p_4 \rangle = V'V' + g^2 \frac{\ln \Lambda}{8\pi^2} \left\{ \frac{3}{2} NV'V' + \frac{3}{2N}(N^2-4)A'A' + \frac{6}{N^2}(N^2-1)AA \right\} + O(g^4).$$

This explicitly shows that the operator Q_1 is not multiplicatively renormalizable. Only certain linear combinations of the O's and the Q's can possibly be multiplicatively renormalized. The above equation will be written in the following simplified manner:

$$V'V' \to V'V' + g^2\lambda \left[\frac{3}{2} NV'V' + \frac{3}{2N}(N^2-4)A'A' + \frac{6}{N^2}(N^2-1)AA \right],$$

with $\lambda = \frac{\ln \Lambda}{8\pi^2}$. We list the results for other matrix elements as follows:

$$VV \to VV + g^2\lambda \left[\frac{3}{2} A'A' \right]$$

$$AA \to AA + g^2\lambda \left[\frac{3}{2} V'V' \right]$$

$$V'V' \to V'V' + g^2\lambda \left[\frac{3}{2} NV'V' + \frac{3}{2N}(N^2-4)A'A' + \frac{6}{N^2}(N^2-1)AA \right]$$

$$A'A' \to A'A' + g^2\lambda \frac{3}{2} \left[NA'A' + \frac{3}{2N}(N^2-4)V'V' + \frac{6}{N^2}(N^2-1)VV \right]$$

$$VA \to VA + g^2\lambda \left[-\frac{3}{2} A'V' \right]$$

$$AV \to AV + g^2\lambda \left[-\frac{3}{2} V'A' \right]$$

$$V'A' \to V'A' + g^2\lambda \left[\frac{3}{2} NV'A' - \frac{3}{2N}(N^2-4)A'V' - \frac{6}{N^2}(N^2-1)AV \right]$$

$$A'V' \to A'V' + g^2\lambda \left[\frac{3}{2} NA'V' - \frac{3}{2N}(N^2-4)V'A' - \frac{6}{N^2}(N^2-1)VA \right]$$

$$SS \to SS + g^2\lambda \left[\frac{3}{N}(N^2-1)SS + \frac{1}{4} T'T' \right]$$

$$PP \to PP + g^2\lambda \left[\frac{3}{N}(N^2-1)PP + \frac{1}{4} T'T' \right]$$

$$TT \to TT + g^2\lambda \left[-\frac{1}{N}(N^2-1)TT + 6(S'S' + P'P') \right]$$

$$S'S' \to S'S' + g^2\lambda \left[\frac{1}{N^2}(N^2-1)TT - \frac{3}{N} S'S' + \frac{1}{4N}(N^2-4)T'T' \right]$$

$$P'P' \to P'P' + g^2\lambda \left[\frac{1}{N^2}(N^2-1)TT - \frac{3}{N} P'P' + \frac{1}{4N}(N^2-4)T'T' \right].$$

$$T'T'\to T'T'+g^2\lambda\Big[\frac{24}{N^2}(N^2-1)\ (SS+PP)+\frac{6}{N}(N^2-4)(S'S'+P'P')$$
$$+\frac{1}{N}\ (2N^2+1)T'T'\Big]\quad .$$

$$SP\to SP+g^2\lambda\Big[\frac{3}{N}(N^2-1)SP+\frac{1}{4}E'T'\Big]$$

$$PS\to PS+g^2\lambda\Big[\frac{3}{N}(N^2-1)PS+\frac{1}{4}E'T'\Big]$$

$$ET\to ET+g^2\lambda\Big[-\frac{1}{N}(N^2-1)ET+6(S'P'+P'S')\Big]$$

$$S'P'\to S'P'+g^2\lambda\Big[\frac{1}{N^2}(N^2-1)ET-\frac{3}{N}S'P'+\frac{1}{4N}(N^2-4)E'T'\Big]$$

$$P'S'\to P'S'+g^2\lambda\Big[\frac{1}{N^2}(N^2-1)ET-\frac{3}{N}P'S'+\frac{1}{4N}(N^2-4)E'T'\Big]$$

$$E'T'\to E'\ T'+g^2\lambda\Big[\frac{24}{N^2}(N^2-1)(SP+PS)+\frac{6}{N}(N^2-4)(S'P'+P'S')$$
$$+\frac{1}{N}(2N^2+1)E'T'\Big]$$

From these results it is easy to find the correct combinations which are multiplicatively renormalized. For example, we can take ***SS-PP,*** and find

$$(SS-PP)\to\Big[1\ +g^2\frac{\ln\Lambda}{8\pi^2}\ .\frac{3(N^2-1)}{N}\Big](SS-PP).$$

Since $\langle p_1,\,p_3|O|p_2,\,p_4\rangle=Z_0^{-1}\langle p_1,\ p_3|O_R|p_2,\,p_4\rangle$, we thus get

$$Z_0=\Big[1-g^2\frac{\ln\Lambda}{8\pi^2}\cdot\frac{3(N^2-1)}{\mathrm{N}}+O(g^4)\Big]\ \text{for}\ \ O=O_5-O_6.$$

Then

$$\gamma_0=-\Lambda\frac{\partial}{\partial\Lambda}\ \ln Z_0=g^2\ \text{[illegible]}\ \frac{(N^2-1)}{N}+O(g^4),$$

or, expressed in terms of $g_R^2=g^2+O(g^4)$,

$$\gamma_0=g_R^2\cdot\frac{3}{8\pi^2}\cdot\frac{(N^2-1)}{N}+O(g_R^4).$$

Finally, the characteristic dimension of the operator

$$O=O_5-O_6\ \text{ is }\ d=\frac{1}{2b}\cdot\frac{3}{8\pi^2}\cdot\frac{N^2-1}{N}.$$

The other combinations can be similarly obtained, and their characteristic dimensions determined. They are all listed in Table I of Sec. III. The coefficients $\alpha_i, \beta_i, \gamma_i$ and δ_i $(i=1, 2, 3, 4)$ in the table are given by the following.

Let

$$K_1=\frac{1}{N}[N^2-1+3N+R], \qquad K_2=\frac{1}{N}[N^2-1-3N+R],$$

$$K_3=\frac{1}{N}[N^2-1+3N-R], \qquad K_4=\frac{1}{N}[N^2-1-3N-R],$$

where $R=(4N^4-11N^2+16)^{1/2}$. Two cases must be distinguished: $N>2$ and $N=2$.

(1) First we consider the case $N>2$. Then it is easy to check that $K_1>K_2>K_3>K_4$. Thus the characteristic dimensions for the four combinations of operators $(i=1, 2, 3, 4)$ are all distinct. The relations between $\alpha_i, \beta_i, \gamma_i$ and δ_i are found to be

$$\beta_i=\frac{N[(N^2-1)(3N+4)-K_iN^2]}{4(N^2-1)(N+2)}\cdot\alpha_i, \quad (i=1, 3)$$

$$\beta_i=\frac{N[(N^2-1)(3N-4)-K_iN^2]}{4(N^2-1)(N-2)}\cdot\alpha_i, \quad (i=2, 4)$$

$$\gamma_i=\frac{2(N^2-1)}{N[N^2-1+K_iN]}\cdot\beta_i, \quad (i=1, 2, 3, 4)$$

$$\delta_i=\frac{N[3-3N^2+K_iN]}{24(N^2-1)}\cdot\alpha_i, \quad (i=1, 2, 3, 4).$$

(2) Next we consider the case $N=2$. Then

$$K_1=15/2, \qquad K_2=K_3=3/2, \qquad K_4=-9/2.$$

That is, the characteristic dimensions of the second and the third combinations of operators are degenerate. The relations between $\alpha_i, \beta_i, \gamma_i$ and δ_i are easily found to be

$$\beta_i=\gamma_i=0, \ (i=1, 2), \quad \delta_1=\frac{1}{6}\alpha_1, \quad \delta_2=-\frac{1}{6}\alpha_2,$$

$$\alpha_i=\delta_i=0, \ (i=3, 4), \quad \gamma_3=\frac{1}{2}\beta_3, \quad \gamma_4=-\frac{1}{2}\beta_4.$$

Therefore, for the case $N=2$, the following four combinations are multiplicatively renormalized :

$$(O_5+O_6)+\frac{1}{6}Q_7, \quad (O_5+O_6)-\frac{1}{6}Q_7,$$

$$(Q_5+Q_6)+\frac{1}{2}O_7, \quad (Q_5+Q_6)-\frac{1}{2}O_7.$$

APPENDIX IV

In this appendix we discuss a generalized Fierz reordering theorem for the four-fermion operators that we are interested in. We first consider the Fierztransformation properties of the Lorentz structures of these operators, neglecting their color-gauge-group structures. Using the standard technique for proving the Fierz reordering theorem[15], one can easily show that, for instance,

$$(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4)=\frac{1}{4}(\bar{\psi}_1\psi_4)(\bar{\psi}_3\psi_2)+\frac{1}{4}(\bar{\psi}_1\gamma_5\psi_4)(\bar{\psi}_3\gamma_5\psi_2)$$
$$+\frac{1}{8}(\bar{\psi}_1\sigma_{\mu\nu}\psi_4)(\bar{\psi}_3\sigma_{\mu\nu}\psi_2)+\frac{1}{4}(\bar{\psi}_1\gamma_\mu\psi_4)(\bar{\psi}_3\gamma_\mu\psi_2)$$
$$+\frac{1}{4}(\bar{\psi}_1 i\gamma_\mu\gamma_5\psi_4)(\bar{\psi}_3 i\gamma_\mu\gamma_5\psi_2).$$

This equation will be conveniently abbreviated as

$$SS=\frac{1}{4}(\bar{S}\bar{S}+\bar{P}\bar{P})+\frac{1}{8}\bar{T}\bar{T}+\frac{1}{4}(\bar{V}\bar{V}+\bar{A}\bar{A}),$$

where the bars denote the Fierz-transformed operators. Similarly, one can show that

$$PP=\frac{1}{4}(\bar{S}\bar{S}+\bar{P}\bar{P})+\frac{1}{8}\bar{T}\bar{T}-\frac{1}{4}(\bar{V}\bar{V}+\bar{A}\bar{A})$$

$$TT=3(\bar{S}\bar{S}+\bar{P}\bar{P})-\frac{1}{2}\bar{T}\bar{T}$$

$$VV=(\bar{S}\bar{S}-\bar{P}\bar{P})-\frac{1}{2}(\bar{V}\bar{V}-\bar{A}\bar{A})$$

$$AA=(\bar{S}\bar{S}-\bar{P}\bar{P})+\frac{1}{2}(\bar{V}\bar{V}-\bar{A}\bar{A}).$$

Similar relations involving the remaining four-fermion operators can be obtained by using the following substitution rules in the above results:

$$SS\to SP,\ PP\to PS,\ TT\to E\ T,$$

$$VV\to(iVA),\ AA\to(-iAV).$$

Thus we have, for example,

$$ET=3(\bar{S}\bar{P}+\bar{P}\bar{S})-\frac{1}{2}\bar{E}\bar{T},\ \text{etc.}$$

An entirely similar technique can be applied to the analysis, of the color-gauge-group structures under the Fierz transformations.[16] One can show that,

$$(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4)=\frac{1}{N}(\bar{\psi}_1\psi_4)(\bar{\psi}_3\psi_2)+\frac{1}{2}(\bar{\psi}_1T_A\psi_4)(\bar{\psi}_3T_A\psi_2),$$

(15) See, for example, the book "*Theory of Weak Interactions in Particle Physics*" by Marshak, Riazuddin and Ryan, (Wiley-Interscience, New York 1969) p. 82. Notice that the tensor-tensor current in our definition is $(\bar{\psi}_1\sigma_{\mu\nu}\psi_2)(\bar{\psi}_3\sigma_{\mu\nu}\psi_4)$ with $\mu\neq\nu$ running through 1, 2, 3, 4.

(16) The basic equation in this analysis is the identity $\delta_{\alpha\mu}\delta_{\beta\nu}=\frac{1}{N}\delta_{\alpha\nu}\delta_{\beta\mu}+\frac{1}{2}\sum_A(T_A)_{\alpha\nu}(T_A)_{\beta\mu}$.

$$(\bar{\psi}_1 T_A \psi_2)(\bar{\psi}_3 T_A \psi_4) = \frac{2(N^2-1)}{N^2}(\bar{\psi}_1\psi_4)(\bar{\psi}_3\psi_2) - \frac{1}{N}(\bar{\psi}_1 T_A \psi_4)(\bar{\psi}_3 T_A \psi_2).$$

These equations will be symbolically abbreviated as

$$O = \frac{1}{N}\bar{O} + \frac{1}{2}\bar{Q},$$

$$Q = \frac{2(N^2-1)}{N^2}\bar{O} - \frac{1}{N}\bar{Q}.$$

When we combine all the above results, we find the following reordering relations for the 20 four-fermion operators:

$$O_1 = -\frac{1}{2N}(\bar{O}_1-\bar{O}_2) - \frac{1}{4}(\bar{Q}_1-\bar{Q}_2) + \frac{1}{N}(\bar{O}_5-\bar{O}_6) + \frac{1}{2}(\bar{Q}_5-\bar{Q}_6),$$

$$O_2 = \frac{1}{2N}(\bar{O}_1-\bar{O}_2) + \frac{1}{4}(\bar{Q}_1-\bar{Q}_2) + \frac{1}{N}(\bar{O}_5-\bar{O}_6) + \frac{1}{2}(\bar{Q}_5-\bar{Q}_6),$$

$$Q_1 = -\frac{(N^2-1)}{N^2}(\bar{O}_1-\bar{O}_2) + \frac{1}{2N}(\bar{Q}_1-\bar{Q}_2) + \frac{2(N^2-1)}{N^2}(\bar{O}_5-\bar{O}_6) - \frac{1}{N}(\bar{Q}_5-\bar{Q}_6),$$

$$Q_2 = \frac{(N^2-1)}{N^2}(\bar{O}_1-\bar{O}_2) - \frac{1}{2N}(\bar{Q}_1-\bar{Q}_2) + \frac{2(N^2-1)}{N^2}(\bar{O}_5-\bar{O}_6) - \frac{1}{N}(\bar{Q}_5-\bar{Q}_6),$$

$$O_5 = \frac{1}{4N}(\bar{O}_1+\bar{O}_2) + \frac{1}{8}(\bar{Q}_1+\bar{Q}_2) + \frac{1}{4N}(\bar{O}_5+\bar{O}_6) + \frac{1}{8}(\bar{Q}_5+\bar{Q}_6) + \frac{1}{8N}\bar{O}_7 + \frac{1}{16}\bar{Q}_7,$$

$$O_6 = -\frac{1}{4N}(\bar{O}_1+\bar{O}_2) - \frac{1}{8}(\bar{Q}_1+\bar{Q}_2) + \frac{1}{4N}(\bar{O}_5+\bar{O}_6) + \frac{1}{8}(\bar{Q}_5+\bar{Q}_6) + \frac{1}{8N}\bar{O}_7 + \frac{1}{16}\bar{Q}_7,$$

$$O_7 = \frac{3}{N}(\bar{O}_5+\bar{O}_6) + \frac{3}{2}(\bar{Q}_5+\bar{Q}_6) - \frac{1}{2N}\bar{O}_7 - \frac{1}{4}\bar{Q}_7,$$

$$Q_5 = \frac{(N^2-1)}{2N^2}(\bar{O}_1+\bar{O}_2) - \frac{1}{4N}(\bar{Q}_1+\bar{Q}_2) + \frac{(N^2-1)}{2N^2}(\bar{O}_5+\bar{O}_6) - \frac{1}{4N}(\bar{Q}_5+\bar{Q}_6) + \frac{(N^2-1)}{4N^2}\bar{O}_7 - \frac{1}{8N}\bar{Q}_7,$$

$$Q_6 = -\frac{(N^2-1)}{2N^2}(\bar{O}_1+\bar{O}_2) + \frac{1}{4N}(\bar{Q}_1+\bar{Q}_2) + \frac{(N^2-1)}{2N^2}(\bar{O}_5+\bar{O}_6) - \frac{1}{4N}(\bar{Q}_5+\bar{Q}_6) + \frac{(N^2-1)}{4N^2}\bar{O}_7 - \frac{1}{8N}\bar{Q}_7,$$

$$Q_7 = \frac{6(N^2-1)}{N^2}(\bar{O}_5+\bar{O}_6) - \frac{3}{N}(\bar{Q}_5+\bar{Q}_6) - \frac{(N^2-1)}{N^2}\bar{O}_7 + \frac{1}{2N}\bar{Q}_7.$$

The remaining relations can be obtained by using the substitutions;

$$O_1 \to iO_3,\ O_2 \to -iO_4,\ O_{5,6,7} \to O_{8,9,10},$$

$$Q_1 \to iQ_3,\ Q_2 \to -iQ_4,\ Q_{5,6,7} \to Q_{8,9,10}.$$

From the above relations it is easy to obtain the following identities, which we will call the generalized Fierz reordering theorem:

$$2N(O_5-O_6)=(\bar{O}_1+\bar{O}_2)+\frac{N}{2}(\bar{Q}_1+\bar{Q}_2),$$

$$(O_1+O_2)+\frac{N}{2}(Q_1+Q_2)=2N(\bar{O}_5-\bar{O}_6),$$

$$\frac{N^2}{(N^2-1)}(Q_5-Q_6)=(\bar{O}_1+\bar{O}_2)-\frac{N}{2(N^2-1)}(\bar{Q}_1+\bar{Q}_2),$$

$$(O_1+O_2)-\frac{N}{2(N^2-1)}(Q_1+Q_2)=\frac{N^2}{(N^2-1)}(\bar{Q}_5-\bar{Q}_6),$$

$$(O_1-O_2)-\frac{N}{2(N-1)}(Q_1-Q_2)=(\bar{O}_1-\bar{O}_2)-\frac{N}{2(N-1)}(\bar{Q}_1-\bar{Q}_2),$$

$$(O_1-O_2)+\frac{N}{2(N+1)}(Q_1-Q_2)=-(\bar{O}_1-\bar{O}_2)-\frac{N}{2(N+1)}(\bar{Q}_1-\bar{Q}_2).$$

Checking against Table I one easily sees that the characteristic dimensions of these particular combinations of operators are all invariant under the Fierz transformations. This is also true for the remaining S, P, T type operators, for which the situation is a little more complicated. Referring to the final part of Appendix III, one can show that for the four possible combinations that are multiplicatively renormalized, one has

$$\alpha_i(O_5+O_6)+\beta_i(Q_5+Q_6)+\gamma_i O_7+\delta_i Q_7$$
$$=\alpha_i(\bar{O}_5+\bar{O}_6)+\beta_i(\bar{Q}_5+\bar{Q}_6)+\gamma_i\bar{O}_7+\delta_i\bar{Q}_7$$

for $N>2$ and $i=1$, 3, and

$$\alpha_i(O_5+O_6)+\beta_i(Q_5+Q_6)+\gamma_i O_7+\delta_i Q_7$$
$$=-\alpha_i(\bar{O}_5+\bar{O}_6)-\beta_i(\bar{Q}_5+\bar{Q}_6)-\gamma_i\bar{O}_7-\delta_i\bar{Q}_7$$

for $N>2$ and $i=2$, 4. As for the case $N=2$, the following four combinations are multiplicatively renormalized :

$$(O_5+O_6)+\frac{1}{6}Q_7,\quad (O_5+O_6)-\frac{1}{6}Q_7,\quad (Q_5+Q_6)+\frac{1}{2}O_7,\quad (Q_5+Q_6)-\frac{1}{2}O_7,$$

and their corresponding characteristic dimensions are

$$\frac{45}{4(11-n)},\quad \frac{9}{4(11-n)},\quad \frac{9}{4(11-n)},\quad \frac{-27}{4(11-n)},$$

respectively. Note the degeneracy of the characteristic dimensions for the second and the third combinations. The Fierz reordering relations give rise to

$$(O_5+O_6)+\frac{1}{6}Q_7=(\bar{O}_5+\bar{O}_6)+\frac{1}{6}\bar{Q}_7,$$

$$(O_5+O_6)-\frac{1}{6}Q_7=-\frac{1}{2}\left[(\bar{O}_5+\bar{O}_6)-\frac{1}{6}\bar{Q}_7\right]+\frac{1}{2}\left[(\bar{Q}_5+\bar{Q}_6)+\frac{1}{2}\bar{O}_7\right],$$

$$(Q_5+Q_6)+\frac{1}{2}O_7=\frac{3}{2}\left[(\bar{O}_5+\bar{O}_6)-\frac{1}{6}\bar{Q}_7\right]+\frac{1}{2}\left[(\bar{Q}_5+\bar{Q}_6)+\frac{1}{2}\bar{O}_7\right],$$

$$(Q_5+Q_6)-\frac{1}{2}O_7=-(\bar{Q}_5+\bar{Q}_6)+\frac{1}{2}\bar{O}_7$$

Again the characteristic dimensions are explicitly shown to be invariant under the Fierz transformations.