An Example of 2-Cocycle

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The quantum system of a non-relativistic free particle is presented as a further physical example involving a 2-cocycle. We identify the non-vanishing commutators between the Galileo boost generator and the momentum operator as the anomalous commutators associated with this 2-cocycle.

Decently the mathematics of cochains and cocycles has been introduced as a convenient language to study various anomalous behaviors of gauge theories[1-4]. Several examples have been given in Refs. 3 and 4 to illustrate the meaning of the cocycles in terms of the anomalies in the system under consideration. In this paper we note an additional example of 2-cocycle.

Consider the quantum mechanics of a non-relativistic free particle, for which Galileo invariance is assumed to hold. Under a general Galilean transformation $g \equiv (R, \vec{V}, \vec{a}, b)$, the space-time is transformed according to

$$\vec{x} \to \vec{x}' = R\vec{x} + \vec{V}t + \vec{a}, \qquad t \to t' = t + b, \tag{1}$$

where R is a 3x3 rotation matrix, and the state ψ of the free particle is transformed into

$$\psi \to \psi' = U\psi, \tag{2}$$

where U is a unitary operator. The wave function $\psi'(\vec{x}, t)$ differs from $\psi(\vec{x}', t')$ only by a phase facto?, since

$$\psi'(\vec{x}, t) = \langle \vec{x}, t | \psi' \rangle = \langle \vec{x}, t | U | \psi \rangle = e^{i\alpha_1} \langle \vec{x}', t' | \psi \rangle$$
$$= e^{i\alpha_1} \psi(\vec{x}', t') . \quad (3)$$

Now because of Galileo invariance, ψ and ψ' satisfy the same Schrödinger equation:

$$(i\,\partial_t + \frac{1}{2m}\nabla^2)\psi = 0 = (i\,\partial_t + \frac{1}{2m}\nabla^2)\psi' \quad (4)$$

From (3) and (4), we find α_1 to be given by

$$\alpha_1(\vec{x}, t; g) = -m\vec{V}\cdot(R\vec{x}+\vec{a}) - \frac{1}{2} mV^2(t-b) . \quad (5)$$

We regard α_1 as a 1-cochain, whose coboundary is, by definition[3,4], given by

$$\Delta\alpha_1(\vec{x}, t; g_1, g_2) = \alpha_1(\vec{x}_1, t_1; g_2) - \alpha_1(\vec{x}, t; g_1 g_2) + \alpha_1(\vec{x}, t; g_1), \quad (6)$$

where x_1, t_1 are the Galileo transforms of $\vec{x}$, t under g_1. Inserting the result (5) in Eq. (6), we get

$$\alpha_2(\vec{x}, t; g_1, g_2) \equiv \Delta\alpha_1 = -\frac{m}{2} b_2 V_1^2 - mb_2 R_2 \vec{V}_1 \cdot \vec{V}_2$$
$$+ mR_2 \vec{V}_1 \cdot \vec{a}_2 . \quad (7)$$

It then follows that $\Delta\alpha_2 = \Delta^2\alpha_1 = 0$, and α_2 is a 2-cocycle. Since the RHS of Eq. (7) does not depend on x and t we will write α_2 as $\alpha_2(g_1, g_2)$ from now on.

The physical significance of α_2 is the following. Since by (2) and (3),

$$U(g)\psi(\vec{x}, t) = e^{i\alpha_1(\vec{x},t;g)}\psi(\vec{x}', t') , \quad (8)$$

we have

$$U(g_1)U(g_2)\psi(\vec{x}, t) = e^{i\alpha_1(\vec{x},t;g_1)} U(g_2)\psi(\vec{x}_1, t_1)$$
$$= e^{i\alpha_1(\vec{x},t;g_1)} e^{i\alpha_1(\vec{x}_1,t_1;g_2)} \psi(\vec{x}_{12}, t_{12})$$
$$= e^{i\alpha_1(\vec{x},t;g_1)} e^{i\alpha_1(\vec{x}_1,t_1;g_2)} e^{-i\alpha_1(\vec{x},t;g_{12})}$$
$$U(g_{12})\psi(\vec{x}, t)$$
$$= e^{i\Delta\alpha_1} U(g_{12})\psi(\vec{x}, t) \quad \text{[by (6)]}$$
$$= e^{i\alpha_2} U(g_{12})\psi(\vec{x}, t), \quad \text{[by (7)]}$$

and therefore

$$U(g_1)U(g_2) = e^{i\alpha_2(g_1,g_2)} U(g_{12}) . \quad (9)$$

This shows that the operator U(g) provides a ray representation (projective representation) of the Galilean group G, meaning that U(g) is a representation of G up to a phase factor $e^{i\alpha_2}$

Now the presence of a 2-cocycle usually signifies the existence of an anomalous commutator (i.e., the Schwinger term) for the generators of the transformation group[3,4]. In the following we will explicitly show that just such an anomalous commutator exists for our system. Consider the following Abelian subgroup of the Galilean group: $g \equiv (1, \vec{V}, \vec{a}, 0)$. Fcr this subgroup, the 2-cocycle α_2 of Eq. (7) is reduced to

$$\alpha_2(g_1, g_2) = m \vec{V}_1 \cdot \vec{a}_2 . \tag{10}$$

Now we choose

$$g_1 = (1, \vec{V}, \vec{0}, 0) \quad \text{and} \quad g_2 = (1, \vec{0}, \vec{a}, 0), \tag{11}$$

and easily verify that

$$g_{12} = g_{21} = (1, \vec{V}, \vec{a}, 0) . \tag{12}$$

Then from (9), (11) and (12), we have

$$U(g_1)U(g_2) = e^{i\alpha_2(g_1,g_2)} U(g_{12}) = e^{im\vec{V}\cdot\vec{a}} U(g_{12}),$$

$$U(g_2)U(g_1) = e^{i\alpha_2(g_2,g_1)} U(g_{21}) = U(g_{21}) = U(g_{12}) \tag{13}$$

and hence

$$U(g_1)U(g_2) = e^{im\vec{V}\cdot\vec{a}} U(g_2)U(g_1) \tag{14}$$

In terms of the generators of the Galilean group ($\vec{K}$ for boosts and $\vec{p}$ for translations), Eq. (14) becomes

$$e^{i\vec{K}\cdot\vec{V}} e^{i\vec{p}\cdot\vec{a}} e^{-i\vec{K}\cdot\vec{V}} e^{-i\vec{p}\cdot\vec{a}} = e^{im\vec{V}\cdot\vec{a}} , \tag{15}$$

or,

$$[\vec{p}\cdot\vec{a}, \vec{K}\cdot\vec{V}] = im\vec{V}\cdot\vec{a}, \tag{16}$$

i.e.,

$$[p_i, K_j] = im\, \delta_{ij}. \tag{17}$$

Eq. (17) is, an anomalous commutator, in the sence that the boost and the translation commute with each other at the group level (see Eq. 12), but not at the algebra level (see Eq. 16).

We also note that a similar commutator

$$[p_i, K_j] = i\, p_0\, \delta_{ij} \tag{18}$$

for the case of Poincare (instead of Galileo) invariance is not anomalous. To see this we take a Lorentz boost and a spatial translation both along the x-axis and find that

$$\begin{pmatrix} x \\ t \end{pmatrix} \xrightarrow[V]{} \begin{pmatrix} \gamma(x+Vt) \\ \gamma(t+Vx) \end{pmatrix} \xrightarrow[a]{} \begin{pmatrix} \gamma(x+Vt)+a \\ \gamma(t+Vx) \end{pmatrix} \approx \begin{pmatrix} \gamma x+Vt+a \\ \gamma t+Vx \end{pmatrix} , \tag{19}$$

$$\begin{pmatrix} x \\ t \end{pmatrix} \xrightarrow[a]{} \begin{pmatrix} x+a \\ t \end{pmatrix} \xrightarrow[V]{} \begin{pmatrix} \gamma(x+a+Vt) \\ \gamma(t+Vx+Va) \end{pmatrix} \approx \begin{pmatrix} \gamma x+a+Vt \\ \gamma t+Vx+Va \end{pmatrix} , \tag{20}$$

neglecting terms of $O(v^3)$ and $O(av^2)$. Comparing (19) and (20), we expect $e^{iK_1 V} e^{ip_1 a}$ and $e^{ip_1 a} e^{iK_1 V}$ to differ by a multiplicative factor describing time translation with an amount (Va):

$$e^{iK_1 V} e^{ip_1 a} e^{-iK_1 V} e^{-ip_1 a} = e^{i(Va)p_0} , \tag{21}$$

which is exactly what the commutator (18) implies. Thus $e^{i\vec{K}\cdot\vec{V}}$ and $e^{i\vec{p}\cdot\vec{a}}$ (together with $e^{ip_0 b}$ and $e^{i\vec{J}\cdot\vec{\omega}}$) provide a true representation of the Poincare group, in contrast to the ray representation of the Galilean group in the previous example.

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