Growth Index with the Cosmological Constant

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(Received July 13, 2011)

We obtain the exact analytical form of the growth index at the present epoch ($a = 1$) in a flat universe with a cosmological constant (*i.e.*, dark energy with the equation of state $\omega_{\rm de} = -1$). For the cosmological constant, we obtain the exact value of the current growth index parameter $\gamma = 0.5547$, which is very close to the well known value 6/11. We also obtain the exact analytic solution of the growth factor for $\omega_{\rm de} = -1/3$ or -1. We investigate the growth index and its parameter at any epoch with this exact solution. In addition to this, we are able to find the exact $\Omega_{\rm m}^0$ dependence of those observable quantities. The growth index is quite sensitive to $\Omega_{\rm m}^0$ at $z = 0.15$, where we are able to use the 2 degree field (2dF) observations. If we adopt the 2dF value of the growth index, then we obtain the constraint $0.11 \leq \Omega_{\rm m}^0 \leq 0.37$ for the cosmological constant model. Especially, the growth index is quite sensitive to $\Omega_{\rm m}^0$ around $z \leq 1$. We might be able to obtain interesting observations around this epoch. Thus, the analytic solution for this growth factor provides a very useful tool for future observations to constrain the exact values of observational quantities at any epoch related to the growth factor for $\omega_{\rm de} = -1$ or $-1/3$.

PACS numbers: 95.36.+x

I. GROWTH INDEX FOR $\omega = -1/3$ AND -1

The background evolution equations in a flat Friedmann-Robertson-Walker universe (with $\rho_{\rm m} + \rho_{\rm de} = \rho_{\rm cr}$) are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_{\rm m} + \rho_{\rm de}) = \frac{8\pi G}{3}\rho_{\rm cr}\,, \tag{1}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -8\pi G\omega_{\rm de}\rho_{\rm de}\,, \tag{2}$$

where $\omega_{\rm de}$ is the equation of state (eos) of dark energy, $\rho_{\rm cr}$ is the critical density, and $\rho_{\rm m}$ and $\rho_{\rm de}$ are the energy densities of the matter and the dark energy, respectively. We consider constant $\omega_{\rm de}$ only. The dark energy does not participate directly in cluster formation, but it alters the cosmic evolution of the background. The linear density perturbation of the matter ($\delta = \delta\rho_{\rm m}/\rho_{\rm m}$) for sub-horizon scales is governed by [1]

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_{\rm m}\delta\,. \tag{3}$$

The textbook solution for the growing mode solution of Eq. (3) for $\omega_{\rm de} = -1$ or $-1/3$ is [2–4]

$$\delta_{\rm g}(a) = \frac{5\Omega_{\rm m}^0}{2}\frac{H(a)}{H_0}\int_0^a X^{-3/2}(a')da' \,, \tag{4}$$

where $X(a) = (aH/H_0)^2 = \Omega_{\rm m}^0 a^{-1} + \Omega_{\rm de}^0 a^{-1-3\omega_{\rm de}}$, here $\Omega_{\rm m}^0 = \rho_{\rm m}^0/\rho_{\rm cr}^0$ and $\Omega_{\rm de}^0 = 1-\Omega_{\rm m}^0$ are the present energy density contrast of the matter and the dark energy, respectively. The growth index f is defined as

$$f = \frac{d\ln\delta_{\rm g}}{d\ln a} \equiv \Omega_{\rm m}(a)^{\gamma} \,. \tag{5}$$

Thus, f is expressed from Eq. (4) as

$$f(\Omega_{\rm m}^0, \omega_{\rm de}, a) = -\frac{3}{2} - \frac{3\omega_{\rm de}(1-\Omega_{\rm m}^0)a^{-1-3\omega_{\rm de}}}{2X} + \frac{aX^{-3/2}(a)}{\int_0^a X^{-3/2}(a')da'} \,. \tag{6}$$

We obtain the exact analytic form of the growth index for $\omega_{\rm de} = -1$ or $-1/3$ at the present epoch ($a=1$, $X(a=1)=1$) as

$$f(\Omega_{\rm m}^0, \omega_{\rm de}, 1) = -\frac{3}{2} - \frac{3}{2}\omega_{\rm de}(1-\Omega_{\rm m}^0) - 3\omega_{\rm de}(\Omega_{\rm m}^0)^{\frac{3}{2}}\frac{\Gamma[1-\frac{5}{6\omega_{\rm de}}]/(\Gamma[\frac{-5}{6\omega_{\rm de}}]\Gamma[1])}{F[\frac{3}{2},\frac{-5}{6\omega_{\rm de}},1-\frac{5}{6\omega_{\rm de}},-\frac{\Omega_{\rm de}^0}{\Omega_{\rm m}^0}]} \,, \tag{7}$$

where Γ is the gamma function and F is the hypergeometric function. We emphasize that this formula is correct only for $\omega_{\rm de} = -1/3$ or -1. For $\Omega_{\rm m}^0 = 1$ one gets $f=1$ for all a, which is consistent with $\delta \propto a$. For $\omega_{\rm de} = -\frac{1}{3}$, f is matched with a non-flat space without the cosmological constant $f(a=1) = -1-\Omega_{\rm m}^0/2 + 5/2(\Omega_{\rm m}^0)^{3/2}/F[3/2,5/2,7/2,1-(\Omega_{\rm m}^0)^{-1}]$ as shown in Ref. [5]. For the cosmological constant, the above formula becomes

$$f(\Omega_{\rm m}^0, -1, 1) \equiv f_L^0 = -\frac{3}{2}\Omega_{\rm m}^0 + 3(\Omega_{\rm m}^0)^{\frac{3}{2}}\frac{\Gamma[\frac{11}{6}]/\Gamma[\frac{5}{6}]}{F[\frac{3}{2},\frac{5}{6},\frac{11}{6},-\frac{\Omega_{\rm de}^0}{\Omega_{\rm m}^0}]} \,, \tag{8}$$

where we use $\Gamma[1]=1$. From the exact form of the growth index f_L^0, we are able to obtain the exact present value of the growth index parameter γ_L^0,

$$\gamma_L^0 \equiv \frac{\ln f_L^0}{\ln\Omega_{\rm m}^0} = \frac{\ln\left[-\frac{3}{2}\Omega_{\rm m}^0 + 3(\Omega_{\rm m}^0)^{\frac{3}{2}}\frac{\Gamma[\frac{11}{6}]/\Gamma[\frac{5}{6}]}{F[\frac{3}{2},\frac{5}{6},\frac{11}{6},-\Omega_{\rm de}^0/\Omega_{\rm m}^0]}\right]}{\ln\Omega_{\rm m}^0} \,. \tag{9}$$

In Fig. 1 we show the present value of the growth index f_L^0 and its parameter γ_L^0 as a function of $\Omega_{\rm m}^0$. In Fig. 1a we show the variation of the present growth index for different values of $\Omega_{\rm m}^0$. It is changed from 0.4073 to 0.6028 for $0.2 \le \Omega_{\rm m}^0 \le 0.4$. Thus, there is up to 32% difference in f_L^0 for the different values of $\Omega_{\rm m}^0$. However, this $\Omega_{\rm m}^0$ dependence is decreased by using the parameter γ_L^0. We show this variation of γ_L^0 in Fig. 1b. γ_L^0 is changed from 0.5580 to 0.5524 for $\Omega_{\rm m}^0 = 0.2$ and 0.4, respectively. We obtain $\gamma_L^0 = 0.5547$ for $\Omega_{\rm m}^0 = 0.3$.

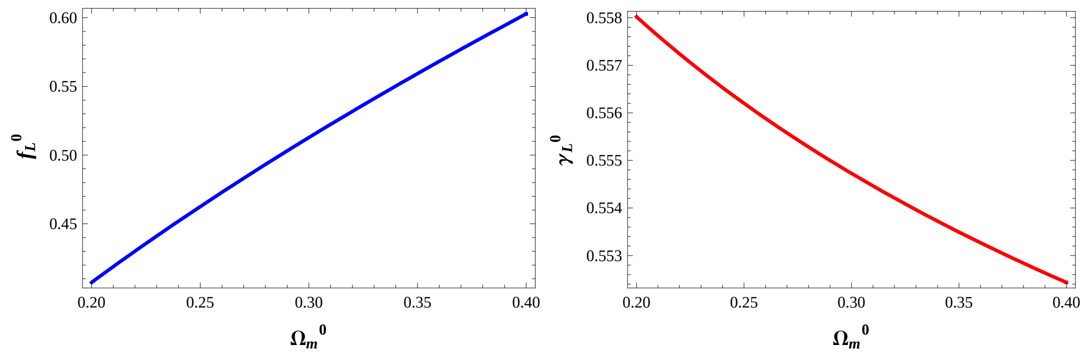

FIG. 1: $\Omega^0_{\rm m}$ dependence of f^0_L and γ^0_L. a) Current value of the growth index f^0_L for $0.2 \leq \Omega^0_{\rm m} \leq 0.4$. b) γ^0_L for the same range of $\Omega^0_{\rm m}$.

II. GROWTH FACTOR FOR $\omega = -1/3$ AND -1

We also obtain the exact analytic solution of the growth factor given in Eq. (4) for $\omega_{\rm de} = -1/3, -1$ (see Appendix for details). The solution of Eq. (A4) for $\omega_{\rm de} = -1$ is

$$\delta^L_{\rm g}(a) = c^L_1 \sqrt{1 + Q a^{-3}} + c^L_2 Q^{\frac{2}{3}} a^{-2} F[1, \frac{1}{6}, \frac{5}{3}, -Q a^{-3}] \, , \tag{1}$$

where $c^L_{1,2}$ are integral constants and $Q = \Omega^0_{\rm m}/\Omega^0_{\rm de}$. We need to determine the c^L_1 and c^L_2 based on the proper initial conditions $\delta^L_{\rm g}(a_i) = a_i$ and $d\delta^L_{\rm g}/da|_{a_i} = 1$ in the matter dominated epoch in order to make this solution become the growing mode solution. We obtain from the initial conditions that $c^L_1 = 1.085464$ and $c^L_2 = -0.943314$ when $\omega_{\rm de} = -1.0$ and $\Omega^0_{\rm m} = 0.3$.

We show the cosmological evolution of the growth factor $\delta^L_{\rm g}$ for different values of $\Omega^0_{\rm m}$ in Fig. 2. The dotted, solid, and dashed lines correspond to $\Omega^0_{\rm m} = 0.4,\ 0.3$, and 0.2 respectively. As $\Omega^0_{\rm m}$ increases, $\delta^L_{\rm g}$ maintains linear growth with a for a longer time, as expected. From the initial conditions we obtain $(c^L_1, c^L_2) = (1.25633, -1.09263)$ and $(0.906875, -0.788711)$ for $\Omega^0_{\rm m} = 0.4$ and 0.2, respectively. We use these values of the coefficients in Fig. 2.

In addition to the present value of the growth index $f(a = 1)$, we are able to obtain the growth index in any epoch from the exact analytic solution of the growth factor (1). For $\omega_{\rm de} = -1$ we find that

$$\begin{aligned} f_L(a) \;&=\; \frac{d \ln \delta^L_{\rm g}(a)}{d \ln a} = \frac{\left(-\frac{3}{2} \frac{A[a]^2 - (c^L_1)^2}{A[a]} - 2B[a] + \frac{3}{10} Q a^{-3} B[a] \frac{F2}{F1} \right)}{(A[a] + B[a])} \, , \\ \text{where} \, , \quad & A[a] = c^L_1 \sqrt{1 + Q a^{-3}} \ , \ B[a] = c^L_2 Q^{\frac{3}{2}} a^{-2} F1 \, , \\ & F1 = F[1, \frac{1}{6}, \frac{5}{3}, -Q a^{-3}] \ , \ F2 = F[2, \frac{7}{6}, \frac{8}{3}, -Q a^{-3}] . \end{aligned} \tag{2}$$

Even though it looks complicated, one is able to obtain $f_L(a)$ by using the solution $\delta^L_{\rm g}$

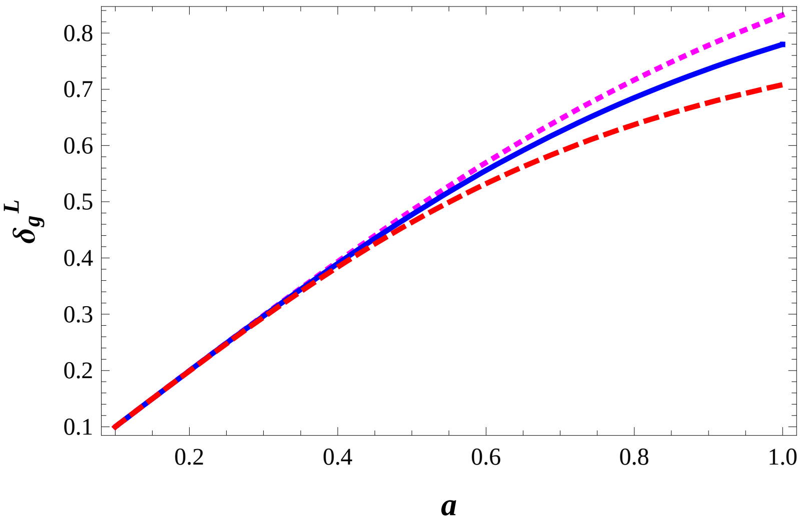

FIG. 2: Cosmological evolution of $\delta_{\rm g}^L$ for $\Omega_{\rm m}^0 = 0.4, 0.3, 0.2$ (from top to bottom).

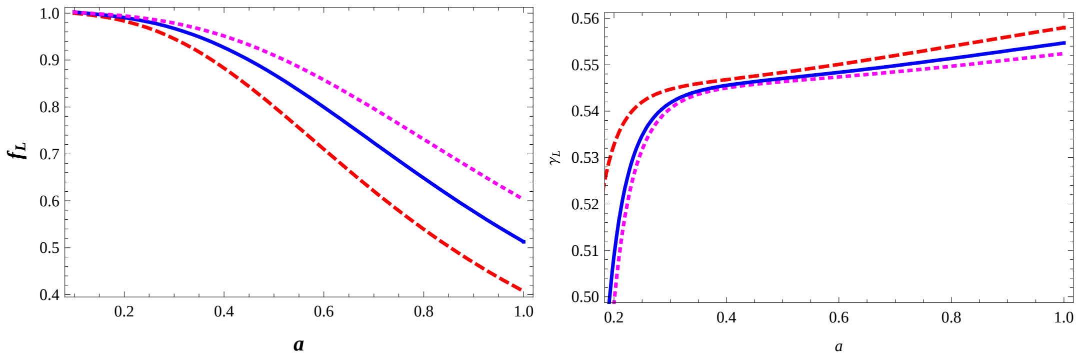

FIG. 3: a) Evolution of $f_L(a)$ for different values of $\Omega_{\rm m}^0$. b) The same for $\gamma_L(a)$. Dotted, solid, and dashed lines correspond to $\Omega_{\rm m}^0 = 0.4, 0.3,$ and 0.2, respectively.

without any difficulty. We are able to find the growth index parameter $\gamma_L(a) = \frac{\ln f_L(a)}{\ln \Omega_{\rm m}(a)}$ in any epoch from this exact analytic form of $f_L(a)$. We also can investigate the $\Omega_{\rm m}^0$ dependence without any ambiguity. Thus, this analytic solution is very useful for the investigation of observational quantities. We show these properties in Fig. 3.

In Fig. 3a, we show the growth index $f_L(a)$ for different values of $\Omega_{\rm m}^0$. The dotted, solid, and dashed lines correspond to the evolution of $f_L(a)$ for $\Omega_{\rm m}^0 = 0.4, 0.3,$ and 0.2, respectively. As we have more matter at present, we have faster growing. Thus, we have bigger values of f_L when we have bigger values of $\Omega_{\rm m}^0$, as shown in Fig. 3a. The $\Omega_{\rm m}^0$ dependence is quite sensitive around $a \simeq 0.87$ ($i.e.$, $z \simeq 0.15$). Thus, the observed value of $f(a = 0.87) = 0.51 \pm 0.15$ from the 2dF galaxy redshift survey will be a good guideline to measure $\Omega_{\rm m}^0$ if the dark energy is the cosmological constant [6]. However, the present value

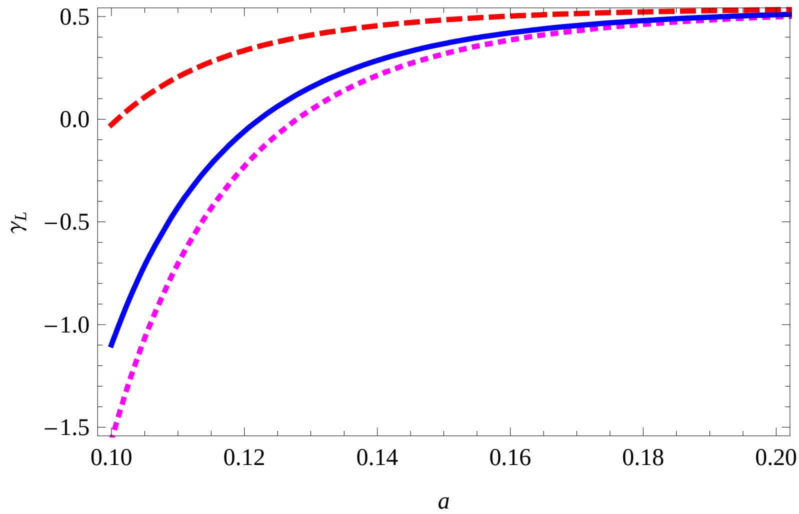

FIG. 4: Cosmological evolution of $\gamma_L(a)$ for $\Omega_{\rm m}^0 = 0.2, 0.3, 0.4$ (from top to bottom).

TABLE I: $\Omega_{\rm m}^0$ is the present value of the matter density contrast, $f_L^{z=0}$ and $f_L^{z=0.15}$ correspond to the growth index at the present and $z = 0.15$, respectively. $\gamma_L^{z=0}$ and $\gamma_L^{z=0.15}$ are the growth index parameters at the present and $z = 0.15$, respectively.

$\Omega_{\rm m}^0$	$f_L^{z=0}$	$\gamma_L^{z=0}$	$f_L^{z=0.15}$	$\gamma_L^{z=0.15}$
0.2	0.407	0.558	0.488	0.555
0.3	0.513	0.555	0.598	0.553
0.4	0.603	0.552	0.685	0.551

of the growth index parameter is insensitive to $\Omega_{\rm m}^0$, as we also obtained in the previous formula. The growth index parameter is sensitive to $\Omega_{\rm m}^0$ around $a \geq 0.8$ (*i.e.*, $z \leq 0.25$), as shown in Fig. 3b. Again, the dashed, solid and dotted lines correspond to the cosmological evolution of $\gamma_L(a)$ for $\Omega_{\rm m}^0 = 0.2, 0.3$, and 0.4, respectively. Especially, the growth index parameter changes dramatically for $a \leq 0.3$ (*i.e.*, $z \geq 2.3$). We show this in Fig. 4. The dashed, solid and dotted lines (from top to bottom) correspond to the evolution of $\gamma_L(a)$ for $\Omega_{\rm m}^0 = 0.2, 0.3$, and 0.4, respectively.

We summarize the results in Table I. As is well known, the growth index parameter γ_L is insensitive to $\Omega_{\rm m}^0$ and a up to $a > 0.3$ [7]. However, $\gamma_L(a)$ shows a strong model dependence around $a \leq 0.3$ (*i.e.*, $z \geq 2.3$). If we naively adopt the 2dF value of f without considering the data error from selection effects, then we obtain the constraint $0.11 \leq \Omega_{\rm m}^0 \leq 0.37$ for the cosmological constant model.

In linear theory, the peculiar velocity $\vec{v}_{\rm pec}$ is related to the peculiar acceleration $\vec{g}$

and/or interior average over-density $\langle\delta\rangle_R$ in a spherical perturbation of radius R, [8]

$$|\vec{v}_{\rm pec}| = \frac{2}{3}\Big|\frac{f\vec{g}}{H\Omega_{\rm m}^0}\Big| = \frac{1}{3}HRf\langle\delta\rangle_R\,. \tag{3}$$

The difference of this peculiar velocity between two different values of $\Omega_{\rm m}^0$ is as large as 29% when we use the different values of $f_L^{z=0.15}$ for $\Omega_{\rm m}^0 = 0.2$ and 0.4. Thus, the exact analytic form of f provides a good analysis tool for galaxy redshift surveys.

APPENDIX A: APPENDIX

From Eq. (6), we need to solve the integration to find the analytic form of f at the present epoch [9],

$$\begin{aligned}\int_0^1 \frac{da}{X^{3/2}(a)} &= \int_0^1 \frac{da}{(\Omega_{\rm m}^0 a^{-1} + \Omega_{\rm de}^0 a^{-1-3\omega_{\rm de}})^{3/2}} = \int_0^1 \frac{da}{(1+\Omega_{\rm de}^0/\Omega_{\rm m}^0 a^{-3\omega_{\rm de}})^{3/2}(\Omega_{\rm m}^0/a)^{3/2}} \\ &= \frac{-1}{3\omega_{\rm de}(\Omega_{\rm m}^0)^{3/2}}\int_0^1 \eta^{-1-5/(6\omega_{\rm de})}(1-r\eta)^{-3/2}d\eta \\ &= \frac{-1}{3\omega_{\rm de}(\Omega_{\rm m}^0)^{3/2}}\frac{\Gamma[\frac{-5}{6\omega_{\rm de}}]\Gamma[1]}{\Gamma[1-\frac{5}{6\omega_{\rm de}}]}F\left[\frac{3}{2},\frac{-5}{6\omega_{\rm de}},1-\frac{5}{6\omega_{\rm de}},-\frac{\Omega_{\rm de}^0}{\Omega_{\rm m}^0}\right]\,,\end{aligned} \tag{A1}$$

where we use $a^{-3\omega_{\rm de}} = \eta$ and $r = -\Omega_{\rm de}^0/\Omega_{\rm m}^0 = 1-(\Omega_{\rm m}^0)^{-1}$. Note that Eq. (A1) is valid for any value of $\omega_{\rm de}$. We can also check the result of a non-flat universe without the cosmological constant in Ref. [5]. In this case we have $k/H_0^2 \equiv \Omega_K = 1-\Omega_{\rm m}^0$. Then we obtain the expression for $X = \Omega_{\rm m}^0 a^{-1} + \Omega_K$. Mathematically, this is the same as the dark energy case with the equation of state of the dark energy $\omega_{\rm de} = -1/3$. Then the above integral (A1) becomes

$$\int_0^1 \frac{da}{X^{3/2}(a)} = \frac{1}{(\Omega_{\rm m}^0)^{3/2}}\frac{\Gamma[\frac{5}{2}]\Gamma[1]}{\Gamma[\frac{7}{2}]}F\left[\frac{3}{2},\frac{5}{2},\frac{7}{2},-\frac{\Omega_K}{\Omega_{\rm m}^0}\right] = \frac{2}{5}(\Omega_{\rm m}^0)^{-3/2}F\left[\frac{3}{2},\frac{5}{2},\frac{7}{2},1-(\Omega_{\rm m}^0)^{-1}\right]\,. \tag{A2}$$

Thus, we can reproduce $f = -1-\Omega_{\rm m}^0/2 + 5/2(\Omega_{\rm m}^0)^{3/2}/F[\frac{3}{2},\frac{5}{2},\frac{7}{2},1-(\Omega_{\rm m}^0)^{-1}]$ in the case of a non-flat space without the cosmological constant.

It is better for us to rewrite the linear density perturbation equation (3) with changing the variable from t to a to get [4]

$$\frac{d^2\delta}{da^2} + \left(\frac{d\ln H}{da} + \frac{3}{a}\right)\frac{d\delta}{da} - \frac{4\pi G\rho_{\rm m}}{(aH)^2}\delta = 0\,. \tag{A3}$$

When we put the growing mode solution (4) into Eq. (3), we get

$$\ddot{\delta}_{\rm g}^{\rm ex} + 2H\dot{\delta}_{\rm g}^{\rm ex} - 4\pi G\rho_{\rm m}\delta_{\rm g}^{\rm ex} - \Big[4\pi G(1+\omega_{\rm de})(1+3\omega_{\rm de})\rho_{\rm de}\Big]\delta_{\rm g}^{\rm ex}\,. \tag{A4}$$

Note that we have changed the notation from $\delta_{\rm g}$ to $\delta_{\rm g}^{\rm ex}$ to reflect the fact that $\delta_{\rm g}^{\rm ex}$ is the solution of Eq. (A4) for any value of $\omega_{\rm de}$. In particular, when $\omega_{\rm de} = -1$ or $-1/3$, the extended solution $\delta_{\rm g}^{\rm ex}$ reduces to $\delta_{\rm g}$. After replacing new parameters $Y = (\Omega_{\rm m})/(\Omega_{\rm de}) = (\Omega_{\rm m}^0)/(\Omega_{\rm de}^0) a^{3\omega_{\rm de}}$ in Eq. (A4), we have

$$Y \frac{d^2 \delta_{\rm g}^{\rm ex}}{dY^2} + \Big[1 + \frac{1}{6\omega_{\rm de}} - \frac{1}{2(Y+1)} \Big] \frac{d\delta_{\rm g}^{\rm ex}}{dY} - \Big[\frac{1}{6\omega_{\rm de}^2 Y} + \frac{3\omega_{\rm de} + 4}{6\omega_{\rm de} Y (Y+1)} \Big] \delta_{\rm g}^{\rm ex} = 0 \,. \tag{A5}$$

Now we try $\delta_{\rm g}^{\rm ex}(Y) = c_1 \delta_1(Y) + c_2 \delta_2(Y) = c_1 Y^i (1+Y)^j + c_2 Y^k B(Y)$, where $c_{1,2}$ are integral constants, because it is the most general combination of the solutions for the above Equation (A5). We obtain the constraints for i and j from δ_1,

$$i = -\frac{1+\omega_{\rm de}}{2\omega_{\rm de}} \,, \quad j = \frac{1}{2} \,. \tag{A6}$$

We also put δ_2 into the above equation (A5) to get

$$\begin{aligned} & Y(1+Y) \frac{d^2 B}{dY^2} + \left[\frac{5}{6\omega_{\rm de}} + \frac{5}{2} + \Big(3 + \frac{5}{6\omega_{\rm de}} \Big) Y \right] \frac{dB}{dY} + \Big(1 + \frac{5}{6\omega_{\rm de}} \Big) B = 0 \,, \\ & \text{when} \quad k = 1 + \frac{1}{3\omega_{\rm de}} \,. \end{aligned} \tag{A7}$$

There are two alternative ways to make the above equation into a hypergeometric differential equation, $Y = -X$ or $1 + Y = X$ [10]. We choose the first case which shows the proper behavior. Now the above Equation (A7) becomes the so-called hypergeometric differential equation,

$$(X)(1-X) \frac{d^2 B}{dX^2} + \left[\frac{1}{2} - \frac{18\omega_{\rm de} + 5}{6\omega_{\rm de}} X \right] \frac{dB}{dX} - \Big(1 + \frac{5}{6\omega_{\rm de}} \Big) B = 0 \,, \tag{A8}$$

with the solution $B(X) = F[1, 1 + \frac{5}{6\omega_{\rm de}}, \frac{5}{2} + \frac{5}{6\omega_{\rm de}}, X]$ being a hypergeometric function [10]. Thus, the full extended solution becomes

$$\delta_{\rm g}^{\rm ex}(Y) = c_1 Y^{-\frac{1+\omega_{\rm de}}{2\omega_{\rm de}}} \sqrt{1+Y} + c_2 Y^{\frac{1+3\omega_{\rm de}}{3\omega_{\rm de}}} F[1, 1 + \frac{5}{6\omega_{\rm de}}, \frac{5}{2} + \frac{5}{6\omega_{\rm de}}, -Y] \,. \tag{A9}$$

Writing $Y = a^{3\omega_{\rm de}} Q$, $Q = \Omega_{\rm m}^0/\Omega_{\rm de}^0$, for $\omega_{\rm de} = -1$ and $-1/3$, we have

$$\delta_{\rm g}^{\rm ex}(a)_{\omega_{\rm de}=-1} = \delta_{\rm g}^L(a) = c_1^L \sqrt{1+Qa^{-3}} + c_2^L Q^{\frac{2}{3}} a^{-2} F[1, \frac{1}{6}, \frac{5}{3}, -Qa^{-3}] \,, \tag{A10}$$

$$\delta_{\rm g}^{\rm ex}(a)_{\omega_{\rm de}=-\frac{1}{3}} = c_1 Q a^{-1} \sqrt{1+Qa^{-1}} + c_2 F[1, -\frac{2}{3}, 0, -Qa^{-1}] \,. \tag{A11}$$

The solution (A9) is a mathematical one. We need to further specify this solution in order to make it a physical solution. If we use the fact that the growth factor $\delta \propto a$ in the matter dominated epoch, then we can determine c_1 and c_2 to make the solution as the growing mode solution. We are also able to find the decaying mode solution if we use decaying mode initial conditions in this solution (A9).

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