Exact Analytical Solutions for the (2+1)-Dimensional Generalized Variable-Coefficients Gross-Pitaevskii Equation

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In this paper, we extend the generalized sub-equation method to investigate some exact analytical solutions for the (2+1)-dimensional (2D) generalized variable-coefficients Gross-Pitaevskii equation (GPE). With the help of symbolic computations, we present five general exact analytical solutions of the 2D GPE, which include bright solitons, dark solitons, and Jacobi elliptic function solutions. Nonlinear dynamics of the chirped soliton pulses is also investigated under the different regimes of soliton management.

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I. INTRODUCTION

Ever since the concept of soliton was introduced in the pioneering work of Zabusky and Kruskal [1] to characterize nonlinear solitary waves which derive from the balance between nonlinearity and dispersion in various nonlinear systems, the study of soliton theory has been developed more vigorously and caused a worldwide study. Besides the study of solitons in fields such as fluid dynamics [2], optics [3, 4], elementary particle physics [5], and plasma physics [6], solitons also appear in condensed matter physics [7], superconductivity physics [8], laser physics [9], biophysics [10], etc. In particular, during the past several decades, the prominent experimental and theoretical progress on Bose-Einstein condensates (BECs) [11] has started a new era for the investigations on solitons and nonlinear physics. It is known that the quantum nonlinear equation governing the evolution of condensates, known as the Gross-Pitaevskii equation, is the same as the classical nonlinear Schrödinger equation in form.

In the past several years, there have been lots of techniques for constructing the exact analytical solutions of the one-dimensional GPE with constant coefficients or with variable coefficients [12–16]. For example in Ref. [14], Li *et al.* presented a direct method for obtaining two families of analytical solutions and analyzed the dynamics of a bright soliton, a train of bright solitons, and a dark soliton thoroughly; in Ref. [13], Li found four families of exact soliton-like solutions by an extended sub-equation method, and so on. More recently, searching for some exact analytical solutions of the 2D GPE has aroused the interest of many scholars [17–21]. To our knowledge, for the constant coefficients 2D GPE, Radha *et al.* have generated bright line solitons by employing the Hirota method

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in Ref. [17], and with the help of the extended sub-equation rational expansion method, some exact analytical solutions were constructed by Guo *et al.* [18]. On the other hand, for the 2D GPE with variable coefficients, Zhong *et al.* have constructed exact periodic wave solutions by an improved homogeneous balance principle and an F-expansion technique in Ref. [19], Hu *et al.* [21] derived the ring dark soliton for a 2D GPE by using a transformation method, Liu *et al.* [22] investigated matter-wave solitons in 2D BECs with time-dependent scattering length in a harmonic trap, and so on.

In this paper, we will extend the generalized sub-equation expansion method [13] to investigate some exact analytical solutions for the following (2+1)-dimensional generalized variable-coefficients GPE [17, 21]:

$$\mathrm{i}\frac{\partial\psi}{\partial t}+\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+F(t)(x^2+y^2)\psi+G(t)\psi|\psi|^2+\mathrm{i}H(t)\psi=0, \tag{1}$$

where $\psi=\psi(x,y,t)$ and the functions $F(t)$, $G(t)$, and $H(t)$ stand for the potential, nonlinearity, and gain coefficients, respectively. The study of nonlinear wave propagation for Eq. (1) is of great interest and has a wide range of applications. It was widely applied to the Bose-Einstein condensate system. It should be noted that Refs. [17–19, 21] each consider an exceptional case when we suitably select different variable coefficients for Eq. (1). In addition, when $H(t)=0$, Saito has demonstrated that a matter-wave bright soliton can be stabilized in 2D free space by causing the strength of the interactions to oscillate rapidly between repulsive and attractive [20].

The rest of this paper is organized as follows. In Section II, we extend the generalized sub-equation expansion method to Eq. (1) and obtain a broad class of exact solutions for these models, and then investigate the main features of the various analytical solutions obtained by using direct computer simulations. In Section III, the nonlinear dynamics of the chirped soliton pulses is investigated from the different regimes of soliton management. Finally, some conclusions are given briefly.

II. EXACT ANALYTICAL SOLUTIONS OF THE (2D) VARIABLE COEFFICIENTS GROSS-PITAEVSKII EQUATION

According to the idea of the generalized sub-equation expansion method [13], we assume the solutions of Eq. (1) to have the following general forms,

$$\psi=\left[\frac{A_0(t)+A_1(t)\phi(\xi)+B_1(t)\phi'(\xi)}{1+a_1(t)\phi(\xi)+b_1(t)\phi'(\xi)}\right]\exp[\mathrm{i}\Theta(x,y,t)], \tag{2}$$

where $\xi=p_1(t)x+p_2(t)y+q(t)$, $\Theta(x,y,t)=\lambda_4(t)x^2+\lambda_3(t)y^2+\lambda_2(t)x+\lambda_1(t)y+\lambda_0(t)$, and $A_0(t)$, $A_1(t)$, $B_1(t)$, $a_1(t)$, $b_1(t)$, $p_1(t)$, $p_2(t)$, $q(t)$, $\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, and $\lambda_4(t)$ are real functions of t to be determined; the variable ϕ=$\phi(\xi)$ satisfies

$$\phi'^2(\xi)=\left(\frac{d\phi(\xi)}{d\xi}\right)^2=h_0+h_1\phi(\xi)+h_2\phi^2(\xi)+h_3\phi^3(\xi)+h_4\phi^4(\xi), \tag{3}$$

where $h_i (i = 0, 1, 2, 3, 4)$ are real constants.

As we all know, the solutions of Eq. (2) are on the basis of Eq. (3). Therefore, we firstly derive a series of fundamental solutions by considering different values of h_0, h_1, h_2, h_3, and h_4. Here we only list two families of hyperbolic function solutions, Jacobi elliptic function solutions, and Weierstrass elliptic doubly periodic solutions.

Case 1. When $h_1 = h_3 = 0$, the Jacobi elliptic functions solutions can be derived:

$$\begin{aligned}
\phi_{11}(\xi) &= \sqrt{-\frac{h_2 m^2}{h_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right), \quad h_4<0, \quad h_2>0, \quad h_0 = \frac{h_2^2 m^2(1-m^2)}{h_4(2m^2-1)^2}. \\
\phi_{12}(\xi) &= \sqrt{-\frac{h_2}{h_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{h_2}{2-m^2}}\,\xi\right), \quad h_4<0, \quad h_2>0, \quad h_0 = \frac{h_2^2(1-m^2)}{h_4(2-m^2)^2}. \\
\phi_{13}(\xi) &= \sqrt{-\frac{h_2 m^2}{h_4(m^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{h_2}{m^2+1}}\xi\right), \quad h_4>0, \quad h_2<0, \quad h_0 = \frac{h_2^2 m^2}{h_4(m^2+1)^2},
\end{aligned} \tag{4}$$

where $0 \le m \le 1$ is a modulus.

When $m \to 1$ and $m \to 0$, the Jacobi functions degenerate to the hyperbolic functions and triangular functions respectively, i.e.,

$$\begin{aligned}
&\text{When } m \to 1, \quad \mathrm{sn}(\xi) \to \tanh(\xi), \quad \mathrm{cn}(\xi) \to \mathrm{sech}(\xi), \quad \mathrm{dn}(\xi) \to \mathrm{sech}(\xi). \\
&\text{When } m \to 0, \quad \mathrm{sn}(\xi) \to \sin(\xi), \quad \mathrm{cn}(\xi) \to \cos(\xi), \quad \mathrm{dn}(\xi) \to 1.
\end{aligned} \tag{5}$$

Case 2. When $h_2 = h_4 = 0$, Eq. (3) has the Weierstrass elliptic doubly periodic type solution:

$$\phi_2(\xi) = \wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right), \quad \text{where } h_3 > 0, \quad g_2 = -4\frac{h_1}{h_3}, \quad g_3 = -4\frac{h_0}{h_3}. \tag{6}$$

Case 3. When $h_0 = h_1 = 0$, we can also obtain two solutions which are similar to those of Case 1.

$$\begin{aligned}
\phi_{31}(\xi) &= \frac{4h_2 k\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}{2(1-\Sigma_1) + 2(1+\Sigma_1)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - (1-\Sigma_1+2h_3k)\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}, \\
\phi_{32}(\xi) &= \frac{4h_2 k\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}{2\Lambda_2 - 2(\Sigma_2+k^2)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - (2h_3k+\Lambda)\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2},
\end{aligned} \tag{7}$$

where $h_2 > 0$, $k = \exp(\sqrt{h_2}c_1)$ is an arbitrary constant, and $\Sigma_1 = k^2(4h_2h_4 - h_3^2)$, $\Sigma_2 = (4h_2h_4 - h_3^2)$, $\Lambda = k^2 - \Sigma_2$.

Case 4. When $h_3 = h_4 = 0$, two solutions of Eq. (3) are derived as follows:

$$\begin{aligned}
\phi_{41}(\xi) &= \frac{2\Delta_1 - \left(\Delta_1 + 4\sqrt{h_2}kh_1\right)\left(\operatorname{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2 + \left(2\Delta_1 - 16k^2h_2\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{8k\left(\sqrt{h_2}\right)^3\left(\operatorname{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}, \\
\phi_{42}(\xi) &= \frac{2\Delta_2 - \left(\Delta_2 + 4\sqrt{h_2}kh_1\right)\left(\operatorname{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2 + \left(16h_2 - 2\Delta_2\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{8k\left(\sqrt{h_2}\right)^3\left(\operatorname{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2},
\end{aligned} \tag{8}$$

where h_2 and k are the same as Case 3, and $\Delta_1 = h_1^2 - 4h_0h_2 + 4k^2h_2$, $\Delta_2 = k^2h_1^2 - 4k^2h_0h_2 + 4h_2$.

Substituting Eq. (2) together with Eq. (3) into Eq. (1), we first remove the exponential terms, then collect coefficients of $\phi(\xi)$, $\phi'(\xi)$, x, and y of the resulting system, and separate the real part and imaginary part for each coefficient. We derive a series of ordinary differential equations (ODEs) with respect to $F(t)$, $G(t)$, $H(t)$, $A_0(t)$, $A_1(t)$, $B_1(t)$, $a_1(t)$, $b_1(t)$, $p_1(t)$, $p_2(t)$, $q(t)$, $\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, and $\lambda_4(t)$. For simplification, we omit the ODE system in this paper.

Finally, solving these ODEs with the help of the symbolic computation system–*Maple*, we can obtain five families of analytical solutions of Eq. (1):

Family 1.

$$\psi_1 = A_1(t)\left[\frac{h_3}{4h_4} + \phi(\xi)\right]\exp[\mathrm{i}\Theta(x,y,t)], \tag{9}$$

where

$$\begin{aligned}
A_1(t) &= \pm\sqrt{-\frac{h_4}{G(t)}\left(c_6^2 + \left(\frac{c_3}{\int\Omega^2\mathrm{d}t}\right)^2\right)}\,\Omega, \\
\xi &= \left(c_6x + \frac{c_3}{\int\Omega^2dt}y\right)\Omega - c_5c_6\int\Omega^2dt + \frac{c_3c_4}{\int\Omega^2dt} + c_2, \\
\lambda_3(t) &= \lambda_4(t) + \frac{\Omega^2}{2\int\Omega^2dt}, \quad \lambda_2(t) = c_5\Omega, \quad \lambda_1(t) = \frac{c_4\Omega}{\int\Omega^2dt}, \\
\lambda_0(t) &= \frac{1}{16}\left[8(c_6^2h_2 - c_5^2) - \frac{3c_6^2h_3^2}{h_4}\right]\int\Omega^2\mathrm{d}t + \frac{1}{16}\frac{3c_3^2h_3^2 + 8c_4^2h_4 - 8c_3^2h_2h_4}{h_4\int\Omega^2\mathrm{d}t} + c_1, \\
F(t) &= \lambda_4'(t) + 2\lambda_4(t)^2, \quad H(t) = -\left(\lambda_3(t) + \lambda_4(t) + \frac{A_1'(t)}{A_1(t)}\right), \\
h_1 &= \frac{h_3(4h_2h_4 - h_3^2)}{8h_4^2}, \quad \Omega = \exp\left(-2\int\lambda_4(t)\mathrm{d}t\right),
\end{aligned} \tag{10}$$

and h_0, h_2, h_3, h_4, c_i $(i = 1,2,3,4,5,6)$ are arbitrary constants, with $G(t)$, $\lambda_4(t)$ being arbitrary functions of t.

From Eqs. (9) and (10), if one sets $h_3 = 0$, then $h_1 = 0$. Therefore, from $\phi_{1\mathrm{i}}(\xi)$, we can obtain three Jacobi elliptic function solutions of Eq. (1):

$$\psi_{1\mathrm{i}} = A_1(t)\phi_{1\mathrm{i}}(\xi)\exp[\mathrm{i}\Theta(x,y,t)], \quad (i = 1, 2, 3). \tag{11}$$

From Eq. (11), when $m \to 1$, the bright soliton and the dark soliton for Eq. (1) are obtained:

$$\begin{aligned} \psi_{14} &= A_1(t)\sqrt{-\frac{h_2}{h_4}}\mathrm{sech}\left(\sqrt{h_2}\xi\right)\exp[\mathrm{i}\Theta(x,y,t)], \\ \psi_{15} &= A_1(t)\sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\xi\right)\exp[\mathrm{i}\Theta(x,y,t)], \end{aligned} \tag{12}$$

where

$$\Theta(x,y,t) = \lambda_4(t)x^2 + \left[\lambda_4(t) + \frac{\Omega^2}{2\int\Omega^2 dt}\right]y^2 + \Omega(c_5 x + \frac{c_4}{\int\Omega^2 dt}y) + \frac{c_6^2 h_2 - c_5^2}{2}\int\Omega^2 \mathrm{d}t + \frac{c_4^2 - c_3^2 h_2}{2\int\Omega^2 \mathrm{d}t} + c_1,$$

and $A_1(t), \xi, F(t), H(t), G(t), \Omega, \lambda_4(t)$ are determined by Eq. (10).

Family 2.

$$\psi_2 = \pm\frac{\Omega}{2}\sqrt{\frac{1}{G(t)}\left(1 + \left(\frac{c_3}{c_6\int\Omega^2 \mathrm{d}t}\right)^2\right)\left(2c_5^2\delta^2 + c_6^2\left(h_3\delta - 4h_4\right)\right)}\frac{\phi(\xi)}{1 + \delta\phi(\xi)}\exp[\mathrm{i}\Theta(x,y,t)], \tag{13}$$

where $\{\lambda_1(t),\ \lambda_2(t),\ \lambda_3(t),\ \lambda_4(t),\ \xi,\ F(t),\ H(t),\ \Omega\}$ are determined by Eq. (10), $\{c_3,\ c_5,\ c_6,\ \delta,\ h_3,\ h_4\}$ are arbitrary constants, and

$$\begin{aligned} \lambda_0(t) &= \frac{1}{2}\frac{c_4^2 c_6^2 - c_5^2 c_3^2}{c_6^2\int\Omega^2 \mathrm{d}t} + c_1, \quad h_0 = \frac{1}{4}\left(\frac{h_3}{\delta^3} - \frac{2c_5^2}{c_6^2\delta^2}\right), \\ h_1 &= \frac{h_3}{\delta^2} - 2\frac{c_5^2}{c_6^2\delta}, \quad h_2 = \frac{1}{2}\left(\frac{3h_3}{\delta} - \frac{4c_5^2}{c_6^2}\right). \end{aligned} \tag{14}$$

If $h_2 = h_4 = 0$ in Eq. (14), then $h_0 = -(1/6)(c_5^2/c_6^2\delta^2)$, $h_1 = -(2/3)(c_5^2/\delta c_6^2)$, $h_3 = (4/3)(c_5^2\delta/c_6^2)$. Then, from $\phi_2(\xi)$, we can obtain the Weierstrass elliptic function solution of Eq. (1):

$$\begin{aligned} \psi_{21} = &\pm\frac{\Omega}{2}\sqrt{\frac{1}{G(t)}\left(1 + \left(\frac{c_3}{c_6\int\Omega^2 \mathrm{d}t}\right)^2\right)\left(2c_5^2\,\delta^2 + c_6^2 h_3\delta\right)} \\ &\times\frac{\wp(\sqrt{h_3}\xi/2, 2/\delta^2, 1/2\delta^3)}{1 + \delta\wp(\sqrt{h_3}\xi/2, 2/\delta^2, 1/2\delta^3)}\exp[\mathrm{i}\Theta(x,y,t)]. \end{aligned} \tag{15}$$

Family 3.

$$\psi_3 = B_1(t)\frac{\phi'(\xi)}{1 + \delta\phi(\xi)}\exp[\mathrm{i}\Theta(x,y,t)], \tag{16}$$

where

$$
\begin{aligned}
&B_1(t)=\pm\delta\sqrt{-\frac{1}{G(t)}\left(c_6^2+\left(\frac{c_3}{\int\Omega^2\mathrm{d}t}\right)^2\right)\Omega}, \quad H(t)=-\left(\lambda_3(t)+\lambda_4(t)+\frac{B_1'(t)}{B_1(t)}\right),\\
&h_3=\frac{4h_4}{\delta}, \quad h_1=\frac{2(\delta^2h_2-4h_4)}{\delta^3},\\
&\lambda_0(t)=\left(-c_6^2h_2+6\frac{c_6^2h_4}{\delta^2}-\frac{c_5^2}{2}\right)\int\Omega^2\mathrm{d}t+\left(c_3^2h_2-6\frac{c_3^2h_4}{\delta^2}+\frac{c_4^2}{2}\right)\frac{1}{\int\Omega^2\mathrm{d}t}+c_1,
\end{aligned}
\tag{17}
$$

where $\{\lambda_1(t),\ \lambda_2(t),\ \lambda_3(t),\ \lambda_4(t),\ \xi,\ F(t)\}$ are determined by Eq. (10), and $\{c_1,\ c_3,\ c_4,\ c_5,\ c_6,\ \delta,\ h_0,\ h_2,\ h_4\}$ are arbitrary constants.

If one sets $h_0=h_1=0$ in Eq. (17), then $\Sigma_1=\Sigma_2=0$ in Eq. (5). Therefore, further setting $k=1$ of the $\phi_{31}(\xi)$, the solution ψ_3 is changed into

$$
\psi_{31}=B_1(t)\frac{4h_2^{3/2}\Upsilon}{(2h_2\delta-\Upsilon)(\Upsilon-2h_2\delta+4\delta h_2)}\exp[\mathrm{i}\Theta(x,y,t)],
\tag{18}
$$

where $\Upsilon=\sinh(\sqrt{h_2}\xi)+\cosh(\sqrt{h_2}\xi)$.

Family 4.

$$
\psi_4=A_1(t)\frac{\phi(\xi)}{1+\delta\phi'(\xi)}\exp[\mathrm{i}\Theta(x,y,t)],
\tag{19}
$$

where

$$
\begin{aligned}
&h_0=\frac{1}{\delta^2}, \quad h_3=h_4=0, \quad A_1(t)=\pm\frac{\delta h_2}{2}\sqrt{-\frac{1}{G(t)}\left(c_6^2+\left(\frac{c_3}{\int\Omega^2\mathrm{d}t}\right)^2\right)\Omega},\\
&\lambda_0(t)=-\frac{1}{4}\left(2c_5^2+c_6^2h_2\right)\int\Omega^2\mathrm{d}t+\frac{1}{4\int\Omega^2\mathrm{d}t}\left(2c_4^2+c_3^2h_2\right)+c_1,
\end{aligned}
\tag{20}
$$

and $\{\lambda_1(t),\ \lambda_2(t),\ \lambda_3(t),\ \lambda_4(t),\ \xi,\ F(t),\ H(t)\}$ are determined by Eq. (10), and $\{c_1,\ c_3,\ c_4,\ c_5,\ c_6,\ \delta,\ h_1,\ h_2\}$ are arbitrary constants. From $\phi_{41}(\xi)$, the solution ψ_4 is reduced to

$$
\psi_{41}=\mp A_1(t)\left[\frac{8c_0^2h_2\left(\cosh(\sqrt{h_2}\xi)-\Upsilon\right)-4c_0h_1\sqrt{h_2}+\Delta_1\Upsilon}{\left(8c_0h_2(-1+c_0\delta\cosh(\sqrt{h_2}\xi))-\delta\Delta_1\Upsilon\right)\sqrt{h_2}}\right]\exp[\mathrm{i}\Theta(x,y,t)],
\tag{21}
$$

where Υ and Δ_1 are also consistent with the above-mentioned.

Family 5. When $h_0=h_1=h_4=0$, we can derive $\phi(\xi)=-h_2/h_3\mathrm{sech}^2(\sqrt{h_2}\xi/2)$ from $\phi_{31}(\xi)$ with $\Sigma_1=-1$. Then the following solution to Eq. (1) can be derived:

$$
\psi_5=A_1(t)\frac{\left[-h_2+\sigma h_2^{3/2}\tanh(\frac{\sqrt{h_2}\xi}{2})\right](\mathrm{sech}\left(\frac{\sqrt{h_2}\xi}{2})\right)^2\sigma}{\delta\left[-h_2+2h_2^2\sigma^2+\sigma h_2^{3/2}\tanh(\frac{\sqrt{h_2}\xi}{2})\right](\mathrm{sech}\left(\frac{\sqrt{h_2}\xi}{2})\right)^2+\sigma h_3}\exp[\mathrm{i}\Theta(x,y,t)],
\tag{22}
$$

where

$$h_3 = \frac{2\delta h_2(1-\sigma^2 h_2)}{\sigma}, \quad A_1(t) = \pm\sqrt{\frac{h_3\delta}{2\sigma G(t)}\left(c_6^2 + \left(\frac{c_3}{\int\Omega^2\mathrm{d}t}\right)^2\right)}\,\Omega,$$
$$\lambda_0(t) = \frac{1}{2}(c_6^2 h_2 - c_5^2)\int\Omega^2\mathrm{d}t + \frac{1}{2\int\Omega^2\mathrm{d}t}(c_4^2 - c_3^2 h_2) + c_1, \tag{23}$$

and $\{\lambda_1(t),\ \lambda_2(t),\ \lambda_3(t),\ \lambda_4(t),\ \xi,\ F(t),\ H(t)\}$ are determined by Eq. (10), and $\{c_1,\ c_3,\ c_4,\ c_5,\ c_6,\ \delta,\ \sigma,\ h_2\}$ are arbitrary constants.

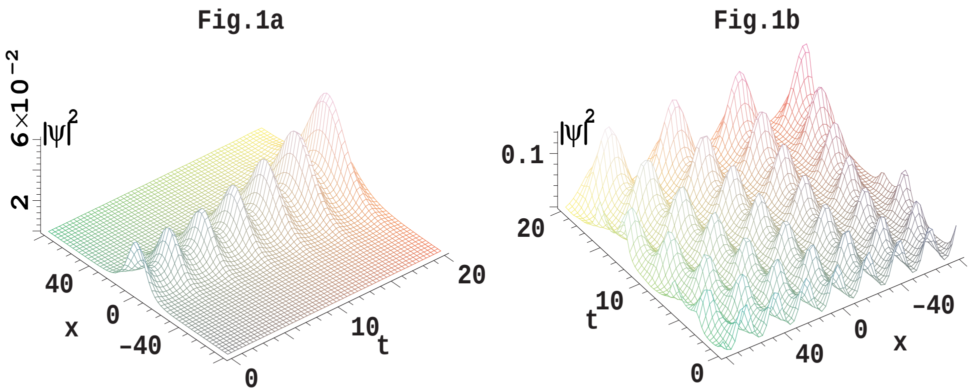


FIG. 1: The evolution plot of the solution given by ψ_{11}; input parameters: $F(t) = 0.0008$, $G(t) = 1 - 0.5\cos(2t)$, $h_2 = 1$, $h_4 = -1$, $c_6 = 0.1$, $c_3 = 1$, $c_4 = 1$, $c_2 = 0$, $c_5 = 0.01$, $y = 1$. In Fig. 1a: $m = 1$; in Fig. 1b: $m = 0.9$.

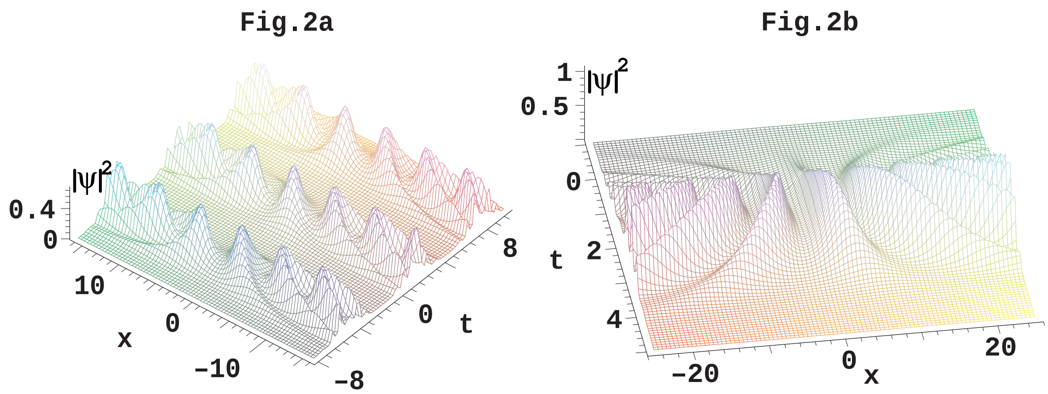


FIG. 2: The evolution plot of the solution given by ψ_{21}; input parameters: $\delta = 1$, $c_3 = 0.1$, $c_6 = 0.1$, $c_4 = 0$, $c_5 = 0.1$, $c_2 = 0$, $y = 1$. In Fig. 2a: $F(t) = -\cos(t)/(\sin(t)+2)$, $G(t) = 1$; in Fig. 2b: $F(t) = -\cos(t)/(\sin(t)+2)$, $G(t) = 1 + 0.8\cos(t)$.

Remark 1: (1) For simplification, we only give five families of solutions of Eq. (1) under some special parameters. In fact, we can also construct other rich exact analytical solutions for Eq. (1) when $\phi(\xi)$ is of other forms. (2) Many known analytical solutions of the GP equation can be recovered from the solutions (11) when limiting $F(t), G(t), H(t)$ to some special parameters, such as, (i) when $H(t) = 0,\ F(t) = \frac{1}{2}\Omega^2(t)$, the one soliton solution in Ref. [17] can be recovered by selecting arbitrary functions and arbitrary constants suitably; (ii) the solutions obtained in [18] can be recovered by setting $G(t) =$ constant, $F(t) = H(t) = 0$, and so on. (3) The analytical solutions obtained in the paper are not ground state solutions. Actually, those solutions correspond to the collective excitation of the condensate, and are usually unstable.

In order to understand the significance of these solutions in Families 1–5, the main soliton features of them are investigated by computer simulations. Here we only consider some cases for some solutions with some special parameters; $|\psi|^2$ denotes the intensity of the solution. Fig. 1 and Fig. 2 present the evolution of the Jacobi elliptic function solutions ψ_{11} and the evolution of the Weierstrass elliptic function solutions ψ_{21}, respectively. From Fig. 1, one can verify that the Jacobi elliptic function $\mathrm{cn}(\xi)$ posses the properties: (i) when $0 < m < 1$, the solution ψ_{11} presents a periodic wave; (ii) when $m = 1$, the solution ψ_{11} reduces to ψ_{14}, which presents the bright-soliton. The time-space evolution of the dispersion Jacobi elliptic periodic wave and bright solitary waves for the solutions $\psi_3,\ \psi_4$, and ψ_5 are shown in Fig. 3, Fig. 4, and Fig. 5.

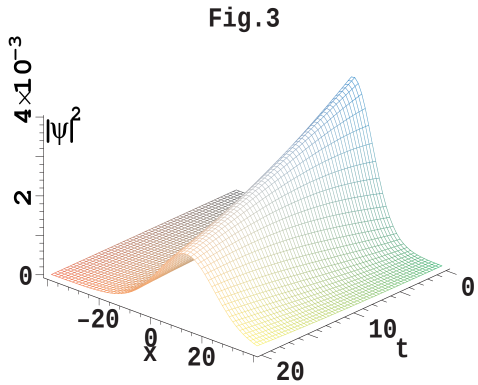


FIG. 3: The evolution plot of the solution given by ψ_{31}. Input parameters: $F(t) = 0.0002,\ G(t) = -1,\ h_2 = 0.25,\ h_3 = -1,\ \delta = 10,\ c_3 = 0.02,\ c_4 = 0.2,\ c_5 = 0.2,\ c_6 = 0.3,\ c_2 = 0,\ y = 1.$

III. DIFFERENT REGIMES OF SOLITON MANAGEMENT

In the following, by virtue of the bright and dark solitons (12), we turn our attention to find solutions for specified soliton management conditions with $F(t) \neq 0$.

(I) Soliton intensity management. Consider Eq. (1) with $G(t) =$ constant, with varying phase modulation, and gain/loss parameters. Suppose that the intensity of the soliton pulse is determined by the relation: $\left(c_6^2 + \left(\frac{c_3}{\int \Omega^2 \mathrm{d}t}\right)^2\right)\Omega^2 = \Phi(t)$, then we can

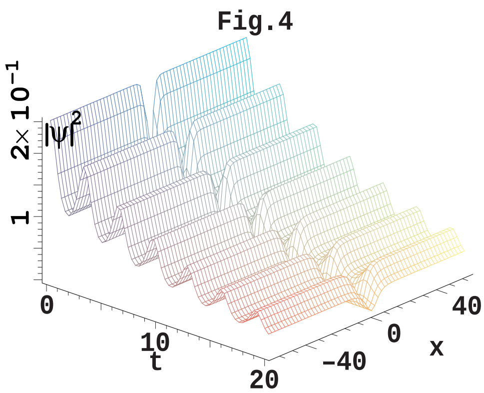
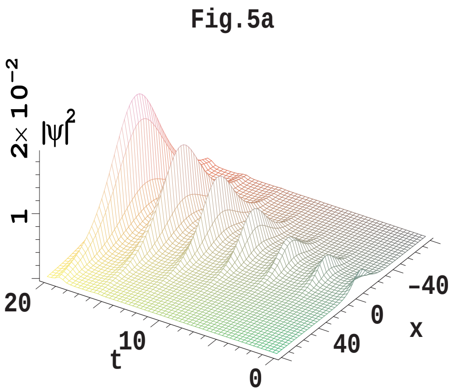


FIG. 4: The evolution plot of the solution given by ψ_{41}. Input parameters: $F(t) = 0.0008,\ G(t) = -1 - 0.9\sin^2(t),\ h_2 = 1,\ c_0 = 2,\ \delta = 2,\ h_1 = 2,\ c_3 = 1,\ c_4 = 0.1,\ c_5 = 0.1,\ c_6 = 1,\ y = 1,\ c_2 = 0.$



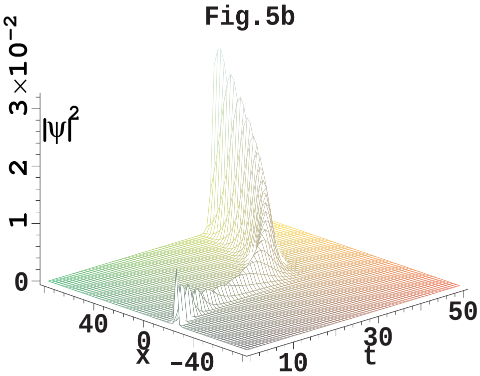


FIG. 5: The evolution plot of the solution given by ψ_5. Input parameters: In Fig. 4a: $F(t) = 1/450,\ G(t) = 100(1 - 0.9\cos^2(t)),\ \delta = 1,\ \sigma = \sqrt{2}/2,\ h_2 = 1,\ c_3 = 1,\ c_5 = 0.3,\ c_6 = 0.1,\ c_4 = 0.2,\ c_2 = 0,\ y = 1$. In Fig. 4b: $F(t) = 1/450,\ G(t) = 100,\ \delta = 1,\ \sigma = \sqrt{2}/2,\ h_2 = 1,\ c_3 = 1,\ c_5 = 3,\ c_6 = 1,\ c_4 = 0.2,\ c_2 = 0,\ y = 1.$

obtain two solutions of Eq. (1) in the form of the chirped dispersion managed solitons (12) with $c_3 = 0$ and $c_6 = 1$, (Note: In what follows, $c_3 = 0$ and $c_6 = 1$ too) and where the main functions Ω, $F(t)$, and $H(t)$ are given by

$$\Omega^2 = \Phi(t), \quad F(t) = \frac{1}{8}(3\Phi'(t)^2 - 2\Phi''(t)\Phi(t))/\Phi(t)^2, \quad H(t) = -\frac{1}{2}\Phi(t) \Bigg/ \int \Phi(t)\mathrm{d}t\,. \quad (24)$$

(II) Soliton pulse width management. Consider Eq. (1) with $G(t) =$ constant, with varying phase modulation and gain/loss parameters. Suppose that the soliton pulse width is determined by the known control function $\Omega = \Theta(t)$, then we can derive two solutions of Eq. (1) in the form of chirped dispersion managed dark and bright solitons (12), where the main coefficients $F(t)$ and $H(t)$ are given by

$$F(t)=\frac{1}{2}(\Theta'(t)^2-\Theta''(t)\Theta(t))/\Theta(t)^2,\quad H(t)=-\frac{1}{2}\Theta(t)^2\Bigg/\int\Theta(t)^2\mathrm{d}t\,. \tag{25}$$

(III) Soliton amplification management. Consider Eq. (1) with $G(t)=$ constant, with varying phase modulation parameters. Suppose that the gain/loss coefficient is determined by the known control $H(t)=\Xi(t)$, then two solutions in the form of (12) can be derived. The main control functions $F(t)$ and Ω are determined by the conditions

$$\begin{aligned}&\Omega=\exp\left[\int\frac{1}{2}(\Xi'(t)-2\Xi(t)^2)\,/\Xi(t)\mathrm{d}t\right],\\&F(t)=\frac{1}{8}(3\Xi'(t)^2-2\Xi''(t)\Xi(t))\Big/\Xi(t)^2+\frac{1}{2}\Xi(t)^2\,.\end{aligned} \tag{26}$$

Now let us consider some examples. Considering some periodical chirped soliton solutions of Eq. (1) from (24). Suppose that the intensity of solitons varies periodically as

$$\left(c_6^2+\left(\frac{c_3}{\int\Omega^2\mathrm{d}t}\right)^2\right)\Omega^2=\Phi(t)=1+\gamma\sin^{2n}t,\quad -1<\gamma\leqslant 1. \tag{27}$$

If $n=1$ in (27), then the main parameters of the soliton solution (12) are as follows:

$$\begin{aligned}&F(t)=\frac{1}{2}\frac{\gamma(2\gamma\sin^2(t)\cos^2(t)-\cos^2(t)+\sin^2(t)+\gamma\sin^4(t))}{(1+\gamma\sin^2(t))^2},\\&H(t)=\frac{1+\sin^2(t)}{\gamma\sin(t)\cos(t)-2\ t-\gamma t},\quad \Omega=1+\gamma\sin^2(t).\end{aligned} \tag{28}$$

If one sets $H(t)=1+\gamma\sin^2 t$, then from (26), the main parameters in the soliton amplification management are

$$\begin{aligned}&\Omega=\exp\left\{-\frac{1}{2}\left[\frac{1}{2}\gamma\sin(2t)-\arctan(\tan(t))\gamma-2t+\ln(1+\gamma\sin^2(t))\right]\right\},\\&F(t)=\frac{\gamma^4\cos^8(t)-4\gamma^3(1+\gamma)\cos^6(t)+\gamma^2(5+12\gamma+6\gamma^2)\cos^4(t)}{2(\gamma\sin^2(t)+1)^2}\\&\qquad-\frac{2\gamma(3+6\gamma+6\gamma^2+2\gamma^3)\cos^2(t)-(1+\gamma)(\gamma^3+3\gamma^2+4\gamma+1)}{2(\gamma\sin^2(t)+1)^2}.\end{aligned} \tag{29}$$

Remark 2. In this part, we considered soliton management for the solutions (12) from different soliton management regimes. Here we only considered the case $c_3=0$ and $c_6=1$ in each regime. In fact, one can consider other c_3 or c_6 values to obtain more rich results.

IV. CONCLUSIONS

In summary, we extend the generalized sub-equation expansion method for constructing some analytical solutions of the (2+1)-dimensional (2D) generalized variable-coefficients Gross-Pitaevskii equation (GPE). With the help of symbolic computation, rich exact analytical solutions of the generalized GPE are obtained, which include bright solitons, dark solitons, Jacobi elliptic function solutions, and Weierstrass elliptic function solutions. From our results, many known results of 2D variable coefficients GPE can be recovered by means of some suitable selections of the arbitrary functions and arbitrary constants. With computer simulations, the main soliton features of the analytical solutions obtained are investigated. Nonlinear dynamics of the chirped soliton pulses is also investigated from the different regimes of soliton management. Our results can be applied to many physical fields, such as Bose-Einstein condensate, nonlinear optics, plasma physics, etc., and opens up opportunities for further studies on relative experiments.

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