A Rich Spectrum of Dynamical Phenomenon in a Forced Parallel LCR Circuit with a Simple Nonlinear Element

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Using a simplified nonlinear element in a second order nonautonomous parallel LCR circuit, we obtain a wide spectrum of dynamics, including chaos with high complexity via torus breakdown. Due to flexibility in choosing the break points in the \((\nu - i)\) characteristic of the nonlinear element, we achieve enlargement of the negative conduction region, large magnitude symmetries of the chaotic attractors, increasing complexity in the dynamical performance, and masking of the large input signal by the chaotic state. Also, through hardware laboratory experiment, transition from quasiperiodic dynamics to chaos, intermittency, reverse period doubling, and period adding phenomena have been observed. The experimental results agree with the results obtained through analytical methods, which are further found to be in very good agreement with the results of a ‘0-1 test’.

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I. INTRODUCTION

Chaotic oscillators have been found to be useful and to have great potential in high performance circuit design for secure communication and telecommunication [1], biomedical engineering applications to human brain and heart [2, 3], and in many others. It is now well known that a second order nonautonomous dissipative nonlinear electronic circuit systems can exhibit chaos through different routes, such as standard period doubling [4, 5], quasiperiodicity, intermittency, period adding [6, 7], Farey sequence, reverse period doubling, antimononicity, and bandmerging [8–10]. Among the different routes, in the case of highly complex dynamical systems, torus breakdown to chaos has been observed in many natural and man-made systems including nonlinear electronic circuits [11, 12]. For instance, torus, and folded torus breakdown to chaos have been observed in a piecewise linear forced LC oscillator with a single diode [13], or with a pair of diodes [14] as a nonlinear element driven by a sinusoidal source. Later, Chua et al. [15] demonstrated the breakdown of a torus to chaos in Chua’s oscillator. Thamilmaran et al. [10] observed chaos in a forced parallel LCR circuit with Chua’s diode as its nonlinear element through a quasiperiodic route. Also, strong chaos via torus breakdown has been observed by them.

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in a forced series $LCR$ circuit with a diode as a nonlinear element [16]. A systematic
design methodology was developed by Chen and his coworkers [17] to create a multifolded
torus and its dynamics including symmetry, bifurcation, eigenspace, and conditions for the
generation of chaos. Very recently, an enriched dynamics along with torus breakdown to
to chaos has been observed in a forced negative conductance parallel $LCR$ circuit with a diode
as the nonlinear element by Manimehan et al. [18]. New scenarios of transition to chaos
through the formation and destruction of multilayered tori in non-invertible maps were
demonstrated by Zhusubaliyev et al. [19]. The chaotic ferroresonance behaviour exhibited
by a non-autonomous circuit which contains a nonlinear flux-controlled inductance was
studied by Meng et al. [20]. Controlling hyperchaos in an array of RCL-shunted Josephson
junctions has been reported by Ilmyong et al. [21]. One of the very recent applications in
this chaotic dynamical study is the hyperchaos encryption using convolutional masking and
model free unmasking, reported by Guo-Yuan et al. [22].

In all these studies, either a negative impedance convertor along with a pn-junction
diode or a pair of pn-junction diodes were used as a nonlinear element, or else a piecewise
linear resistor, namely Chua’s diode, was used as the nonlinear element. Therefore, it will be
quite valuable from the nonlinear dynamics point of view, to construct an electronic circuit
with a simple nonlinear element, which exhibits a wide spectrum of dynamical phenomena.
In this connection, very recently, the present authors constructed a simple nonlinear element
with one op-amp, and three linear resistors, and used the same in a forced series $LCR$ circuit,
to obtain chaos with high complexity, showing higher values of the Lyapunov exponent and
Lyapunov dimension [23]. The advantage of our nonlinear element over the others is that
(i) it has minimal number of circuit elements, (ii) it is mathematically tractable, and (iii)
it exhibits flexibility and a larger range of breakpoints ($\pm B_P$), leading to wider conductance
region. The above leads to chaotic attractors with high magnitude symmetry and also
masks a large input signal and hence is suitable for communicating a wide spectrum of
signals. From this study, it was confirmed that this simple nonlinear element is sufficient
to generate different kinds of chaotic dynamics in nonautonomous circuit systems. It is
understood that, while in a forced series $LCR$ circuit, the dynamics will emanate from a
fixed point, in the case of a forced parallel $LCR$ circuit, it will start from the limit cycle,
indicating the lowest order that can model the onset of oscillations in a dynamical system
and capturing the rich non-periodic oscillations of higher order systems [24]. Motivated by
the above, in the present paper, the chaotic dynamics of a forced parallel $LCR$ circuit with
our simplified nonlinear element is studied, and the results based on our investigations on
a wide spectrum of dynamics including chaos via torus breakdown with high complexity
have been presented.

The plan of the paper is as follows. In Section II, we present the circuit realization
of the proposed nonautonomous circuit. The results of the observations from laboratory
hardware experiment are presented in Section III. The analytical calculations on the dy-
namics of the nonlinear circuit system are presented in Section IV. Section V contains the
results of the ‘0-1 test’. Finally, in Section VI, the results are summarized and concluded.
II. REALIZATION OF THE NONLINEAR ELEMENT AND THE CIRCUIT

The complex dynamics of electronic circuits solely depends on the nonlinear element used in the circuit. While studying the complex dynamical behaviour, the sub-circuit design with the nonlinear element must be simple enough to be constructed and modeled using standard electronic components. Kennedy [25] has given a detailed design methodology for nonlinear elements, which has been effectively used by several authors for designing nonlinear elements with varying components [23, 26–28]. In the present work, the nonlinear element \((N_R)\) in the sub-circuit has been constructed with one op-amp and three linear resistors \((R_1, R_2, R_3)\) as shown in Fig. 1(a), and is used to realize the negative slope of \(G_a = -0.56 \text{ mS}\), and a positive slope of \(G_b = 2.5 \text{ mS}\), with break points \(\pm B_p = 3.8 \text{ V}\) (see Fig. 1(b)). The three piecewise linear regions in the characteristic curve (Fig. 1(b)) are denoted as \(D_0\), \(D_+\), and \(D_-\). Unlike the present case, break points in other autonomous and nonautonomous circuits [25, 28] are to be controlled normally, because otherwise, they will be coming out of the negative conductance region. This particular aspect is considered to be one of the major advantages of our nonlinear element.

The circuit realization of a forced parallel \(LCR\) circuit, to which the proposed simplified nonlinear element \((N_R)\) given in Fig. 1(a) is connected in parallel as shown in Fig. 2. In the circuit, a capacitor \(C\), an inductor \(L\), a resistor \(R\), and an external signal \(f(t) = F \sin \Omega t, \Omega = 2\pi\nu\), and the proposed simplified nonlinear element \((N_R)\) are connected in parallel. By applying Kirchhoff’s laws to the circuit, the state equations are written in terms of the following set of two first-order coupled nonautonomous differential equations.

\[
C \frac{dv}{dt} = \frac{1}{R} (F \sin \Omega t - v) - i_L - g(v),
\]

\[
L \frac{di_L}{dt} = v.
\]

Here, \(v\) and \(i_L\) denote the voltage across the capacitor \(C\) and the current flowing through the inductor \(L\), respectively. \(i_C\) is the current flowing through the capacitor, \(i_N\) is the current through the nonlinear element. The amplitude of the external signal \(F\) is treated as the control parameter. The mathematical form of the piecewise linear function \(g(v)\) is given by

\[
g(v) = \begin{cases} 
G_b v + (G_a - G_b) & v \geq B_p \\
G_a v & |v| \leq B_p \\
G_b v - (G_a - G_b) & v \leq -B_p
\end{cases}
\]

where \(G_a\) and \(G_b\) are the values of the negative and positive slopes in the \((v-i)\) characteristic curve of the nonlinear element (see Fig. 1(b)), respectively.
FIG. 1: (a) Realization of the nonlinear element ($N_R$), with one op-amp ($\mu$A741IC), supply voltage $\pm 9$ V and three linear resistors, $R_1$, $R_2$, and $R_3$, and (b) ($v-i$) characteristic curve of the nonlinear element with a negative slope of $G_a = -0.56$ mS, a positive slope of $G_b = 2.5$ mS, and $\pm B_p = 3.8$ V represent the break points in the characteristic curve at which the negative conductance changes into positive conductance. $D_0$ represents the central negative conduction region and $D_{\pm}$ and are the two outer positive conduction regions.

III. ENRICHED BIFURCATION PHENOMENA: EXPERIMENTAL OBSERVATION

We carried out an experiment to study the enriched dynamics of the electronic circuit given in Fig. 2 by fixing the following circuit parameters as $C = 13.13$ nF, $L = 163.6$ mH,
FIG. 2: Circuit realization of the forced parallel LCR circuit with the simplified nonlinear element $N_R$ and a sinusoidal signal $f(t) = F \sin \Omega t$ and $\Omega = 2\pi \nu$. The values of the circuit elements are chosen as $C = 13.13$ nF, $L = 163.6$ mH, $R = 2.05$ k, and the frequency of the external signal is chosen as $\nu = 1.678$ kHz.

$R = 2.05$ k, and the frequency of the external periodic signal as $\nu = 1.678$ kHz and its amplitude ($F$) is treated as a control parameter. The nonlinear element subcircuit is constructed with one op-amp ($\mu$A741IC) with a supply voltage of ±9 V and three linear resistors. The values of three resistances in the nonlinear element sub-circuit (Fig. 1(a)) are chosen as $R_1 = 1.990$ k, $R_2 = 1.981$ k, and $R_3 = 1.989$ k. The above values of the circuit parameters chosen here for our experiment are based on the requirement of the slope values $G_a$ and $G_b$ in its ($v-i$) characteristic curve, so as to give the required chaotic dynamics. However, electronic components available “off-the-shelf” are only in standard values and hence the values are slightly tuned based on our requirement. From the ($v-i$) characteristic of the nonlinear element ($N_R$) observed experimentally, the values of $G_a$, $G_b$, and $B_p$ are calculated to be $G_a = -0.56$ mS, $G_b = 2.5$ mS, and $B_p = \pm 3.8$ V. To begin with, we have carried out an experimental study of the dynamics of the circuit given in Fig. 2. When the driving signal is switched off for setting the amplitude $F = 0$ (which corresponds to the autonomous case), a limit cycle can be observed (Fig. 3(a)). To begin the experiment, the driving signal is switched on and its amplitude ($F$) is slowly increased from 0 V and the response of the system is observed. The system is found to progress through a series of transitions from periodic motion to aperiodic motion followed by periodic windows. The system is found to transit from a limit cycle attractor to a quasiperiodic (torus) attractor and then to chaos via torus breakdown, followed by a reverse period doubling route, namely torus, chaos, 8T, 4T, 2T, and 1T periodic oscillations. Apart from the above transitions, period adding, Farey sequences, period doubling, and boundary crisis are also observed. The results of the above investigations are discussed one by one in the following subsections.
III-1. Quasiperiodic route and torus breakdown to chaos

When the control parameter, namely, the amplitude of the external periodic force $F$ is increased from zero, the system evolves from a limit cycle and enters slowly into the quasiperiodic region. For example, when the external forcing signal is absent, a period-1 limit cycle is observed. On increasing the amplitude slowly, at $F = 1.53$ V, a torus is observed. When the ratio between the external driving frequency and the circuit frequency is irrational, the circuit exhibits quasiperiodic motion for all values of the external frequency $\nu$, while confining the driving amplitude ($F$) to the range $F \in (0.15$ V, 2.6 V). Figure 3(a) shows the time plot of the limit cycle attractor, when $F = 0$, and Fig. 3(b) shows the time plot of the quasiperiodic attractor for $F = 1.53$ V and $\nu = 1.678$ kHz. In Figs. 3(a) and 3(b), the upper trace is for the voltage $v(t)$ across the capacitor $C$, and that of the current $i_L(t)$ flowing through the inductor $L$ is in the lower trace.

Figure 4 shows the typical quasiperiodic attractor of (a) phase portrait, (b) its Poincaré map in the ($v_C - i_L$) plane, and (c) the respective power spectrum of $v_C(t)$ for $F = 1.53$ V and $\nu = 1.678$ kHz. Since the attractor is a torus, the Poincaré map is obviously a closed curve. For a fixed frequency of $\nu = 1.678$ kHz, when the amplitude is
increased in the range $F \in (0.5 \text{ V}, 5.0 \text{ V})$, torus breakdown occurs for values of $F \geq 4.75 \text{ V}$, followed by periodic windows and finally ending up with a chaotic attractor. A typical chaotic attractor, its Poincaré map in the $(v_c - i_L)$ plane, and the respective power spectrum of $v_c(t)$ are shown in Figs. 5(a)–5(c) for $F = 4.75 \text{ V}$ and $\nu = 1.678 \text{ kHz}$. Since the attractor is chaotic the Poincaré map is discontinuous, and the broadband nature of the power spectrum is also obvious. Also, when the experiment is repeated with frequency ($\nu$) scanning, the torus breakdown to chaos is again observed. For example, for the specific value of $F = 1.53 \text{ V}$, upon increasing the frequency ($\nu$) from 1.65 kHz to 1.7 kHz, the system exhibits torus breakdown.

### III-2. Period-doubling scenario

Once the torus is destructed into chaos, the system undergoes a ubiquitous period-doubling and period halving or reverse period-doubling scenario. When the amplitude is zero for a fixed frequency $\nu = 1.678 \text{ kHz}$, a period-1 limit cycle is observed. On increasing the amplitude slowly, at $F = 1.53 \text{ V}$, a torus is observed. For a further increase of the amplitude, a period-3T window and then a chaotic attractor are observed at $F = 3.200 \text{ V}$ and $F = 4.779 \text{ V}$ respectively. A reverse period doubling sequence is observed, when the amplitude $F$ is increased further; a stable period-4 limit cycle for $F = 5.940 \text{ V}$, a stable period-2 limit cycle for $F = 6.000 \text{ V}$, and a period-1 limit cycle for $F = 6.200 \text{ V}$. Beyond the reverse period-doubling scenario, a boundary crisis is observed for higher driving amplitudes in the range $F \in (6.210 \text{ V}, 7.000 \text{ V})$. All the above observations are presented in Fig. 6, as
FIG. 5: Experimentally observed chaotic attractor. (a) Phase portrait, (b) its Poincaré surface of section in the ($v_c - i_L$) plane. Horizontal axis 3 ms/div, and vertical axis 5 V/div., and (c) the corresponding power spectrum of $v_c(t)$ for $F = 4.75$ V and $\nu = 1.678$ kHz. Horizontal axis frequency in kHz, and vertical axis amplitude.

projections of the phase portraits in the ($v - i_L$) plane. From Fig. 6, it can be observed that the system is exhibiting torus breakdown to chaos via a 3T window and then reverse period-doubling followed by periodic windows. The simulation results match exactly with our experimental observations thereby qualitatively confirming the results. For the sake of simplicity, we do not include the results in this paper.

III-3. Intermittency route to chaos

Apart from the routes of torus breakdown to chaos and reverse period doubling, the system also exhibits an intermittency route to chaos. This intermittency route to chaos can be observed both experimentally and analytically either by varying the amplitude for fixed frequency or by varying the frequency for fixed amplitude. From the experimental point of view, varying the frequency and fixing the amplitude is a more convenient way to observe the intermittency. Therefore, for a fixed amplitude of $F = 1.561$ V and for the frequency range $\nu \in (1.955$ kHz, 1.962 kHz), a stable period-2 limit cycle disappears through a tangent bifurcation, leading to chaos. However, a section of the period-2 limit cycle still remains, and most of the time the trajectory behaves as if it is approaching a period-2 limit cycle. Intermittent behaviour is observed, when the frequency $\nu$ is increased, that is, the periodic oscillations are interrupted by intermittent amplitude bursts as the frequency is increased from 1.955 kHz to 1.962 kHz. Figure 7(a) shows a time waveform of the stable period-2T orbit for $\nu = 1.956$ kHz. The intermittent period-2T limit cycle for $\nu = 1.960$ kHz and for $\nu = 1.961$ kHz and its respective time waveforms are shown in Figs. 7(b) and 7(c). The chaotic burst in amplitude has been clearly observed for $\nu = 1.962$ kHz in Fig. 7(d).
FIG. 6: Experimentally observed torus and period-doubling scenario at the fixed frequency $\nu = 1.678 \text{ kHz}$. The driving amplitude $F$ is treated as the control parameter and the phase portraits are represented in the $(\nu - i_L)$ plane. (a) torus, $F = 1.619 \text{ V}$; (b) period $3T$ window, $F = 3.20 \text{ V}$; (c) chaos, $F = 4.779 \text{ V}$; (d) period - $4T$, $F = 5.490 \text{ V}$; (e) period - $2T$, $F = 6.00 \text{ V}$; and (f) period - $1T$, $F = 6.20 \text{ V}$. Horizontal axis 3 ms/div and vertical axis 5 V/div.

With a further decrease in the frequency, the system gives birth to fully developed chaos. The average laminar length $\langle l \rangle$ during this transition is found to comply with the law $\langle l \rangle = \epsilon^{-\alpha}$ with $\alpha = 0.57$, where $\epsilon = (\omega_c - \omega)$ and $\omega_c$ is the bifurcation threshold. The variation of the laminar length with difference in frequency is presented as a log-log plot in Fig. 8, and the slope is calculated as $\alpha = 0.57$. Thus, from our experimental results, it is concluded that the system admits an intermittency route to chaos and its existence is confirmed from Fig. 8.
FIG. 7: Type-I intermittent behaviour. Time wave form of voltage $v_c$ across the capacitor $C$ of (a) period-2T orbit for $\nu = 1.956$ kHz, (b) intermittent period-2T limit cycle for $\nu = 1.960$ kHz, (c) intermittent period-2T limit cycle for $\nu = 1.961$ kHz, and (d) frequently intermittent of period-2T limit cycle for $\nu = 1.962$ kHz.

IV. RESULTS OF THE ANALYTICAL STUDY

In order to confirm our experimental results, and to check whether the results are mathematically tractable in this section, we carry out an analytical study of the dynamics of the nonlinear circuit considered. For this, first the state Eqs. (1) and (2) are rewritten by rescaling the variables and parameters as $v = B_p x$, $i_L = GB_p y$, $t = C\tau/G$, $G = 1/R$, and $\Omega = \omega G/C$. The rescaled forms of Eqs. (1) and (2) are given by

$$\frac{dx}{dt} = f \sin \omega t - x - y - g(x),$$

$$\frac{dy}{dt} = \beta x,$$

where $g(x)$ is given by

$$g(x) = \begin{cases} bx + (a - b) & x \geq 1.0, \\ ax & |x| \leq 1.0, \\ bx + (b - a) & x \leq -1.0. \end{cases}$$

While writing Eqs. (3) and (4), $\tau$ is redefined as $t$. In Eq. (5), $a = G_a/G$, $b = G_b/G$, $\beta = C/LG^2$, and $f = B_p F$. The piecewise linear function $g(x)$ is associated with three regions namely $D_0$, $D_+$, and $D_-$. While $D_0$ is the central negative conduction region, $D_\pm$ are the two outer positive conduction regions (see Fig. 1(b)).
IV-1. Linear stability

The piecewise linear function $g(x)$ given in Eq. (5) is symmetric with respect to the origin, so that it is invariant under the transformation

$$(x, y) \rightarrow (X, Y) = (-x, -y).$$

In this case, the equilibrium points can be obtained by making the right hand sides of Eqs. (3) and (4) to be zero at the equilibrium points. It follows from the form of $g(x)$ that Eq. (5) has a unique equilibrium in each of the following three regions.

$$D_+ = \{(x, y) | x \geq 1\},$$

$$D_0 = \{(x, y) | |x| \leq 1\},$$

$$D_- = \{(x, y) | x \leq -1\}.$$  

The equilibrium points $(x_0, y_0)$ in the three regions are explicitly given by

$$P^+ = (0, b-a) \in D_+,$$

$$O = (0, 0) \in D_0,$$

$$P^- = (0, a-b) \in D_-.$$  

FIG. 8: Type I intermittent behaviour: log-log plot of the variation of frequency with laminar length, and the slope is calculated as $\alpha = 0.57$ V.
The stability determining eigenvalues \((\lambda_1, \lambda_2)\) in the region \(D_0\) are calculated from the stability matrix

\[
A_0 = A(\beta, a) = \begin{pmatrix}
-(1 + a) & -1 \\
\beta & 0
\end{pmatrix},
\]

as

\[
\lambda_{1,2} = -(1 + a) \pm \frac{\sqrt{(1 + a)^2 - 4\beta}}{2}.
\]

The eigenvalues \((\lambda_1, \lambda_2)\) depend on the value of the negative slope ‘a’ in the \((v - i)\) characteristic of the nonlinear element and that of \(\beta\) which is determined by the circuit parameters. Upon using the values of the circuit parameters used in the experiment here, in Eq. (14), we obtain a pair of complex conjugate eigenvalues with positive real parts.

\[
\lambda_1 = 0.074 + (0.503664736)i,
\]

\[
\lambda_2 = 0.074 - (0.503664736)i.
\]

This indicates that \((x_0, y_0) = (0, 0)\) is an unstable focus point. As a consequence of unstable focus, the dynamics changes as a control parameter is smoothly varied, that is, sudden qualitative change takes place in the nature of the motion, that is, bifurcation occurs. Upon a further change in the control parameter value, this bifurcation will lead to chaos. Similarly, the stability matrix for the regions \(D^+\) and \(D^-\) is given by

\[
A_\pm = A(\beta, b) = \begin{pmatrix}
-(1 + b) & -1 \\
\beta & 0
\end{pmatrix},
\]

and the eigenvalues \((\lambda_3, \lambda_4)\) are calculated as

\[
\lambda_{3,4} = -(1 + b) \pm \frac{\sqrt{(1 + b)^2 - 4\beta}}{2}.
\]

Upon using the circuit parameter values used in the experiment in Eq. (18), we obtain a pair of negative eigenvalues, and hence the fixed points \(P^+\) and \(P^-\) are stable nodes. Naturally, these fixed points can be observed based on the initial values \(x(0)\) and \(y(0)\) in Eqs. (3) and (4), when \(f = 0\). In fact, Eqs. (3) and (4) can be integrated explicitly in terms of elementary functions in each of the three regions \(D_0\), \(D_\pm\), and the resulting solutions can be matched across the boundaries to obtain the full solution.

One finds that as long as the initial conditions are confined to the region \(D_0\), because of the unstable nature of the fixed point, focus, a stable limit cycle results. However, if the initial conditions are chosen in the regions \(D_\pm\), the system will end up in one of the fixed points \(P^+\) or \(P^-\) as the case may be, since they correspond to stable nodes. When the forcing periodic signal is switched on \((f \neq 0)\), there is an interaction between the limit cycle motion of the system and the external periodic signal, resulting in a bifurcation which gives rise to a quasiperiodic solution as observed in the experimental observations. It may be noted that, in contrast to the above, in the case of the Murali-Lakshmanan-Chua (MLC)
circuit \[6\], for the chosen parameter values, the fixed point \(O\) is hyperbolic, and the fixed points \(P^+\) and \(P^-\) are stable foci \[23\]. Therefore, no limit cycle motion occurs in the case of an MLC circuit. Thus, since the system starts with a limit cycle motion in the autonomous case, \((f = 0)\), it is advantageous to have a rich variety of bifurcation phenomena, in the case of the present parallel \(LCR\) circuit.

IV-2. Explicit analytic solution

(i) Region \(D_0(1)\) :
In this region, \(g(x) = ax\) and hence Eqs. (3) and (4) become
\[
\begin{align*}
\frac{dx}{dt} &= f \sin \omega t - x - y - ax, \quad (19) \\
\frac{dy}{dt} &= \beta x. \quad (20)
\end{align*}
\]
Differentiating Eq. (20) once with respect to time and using Eq. (19) in the resultant equation, we get
\[
\frac{d^2y}{dt^2} + (1 + a) \frac{dy}{dt} + \beta y = \beta f \sin \omega t. \quad (21)
\]
As Eq. (21) is a linear second-order inhomogeneous differential equation with constant coefficients, its solution can be found using the standard method. The form of the solution can be written as
\[
y(t) = A_1 e^{\lambda_1 t} + B_1 e^{\lambda_2 t} + C \sin \omega t + D \cos \omega t, \quad (22)
\]
where \(\lambda_1\) and \(\lambda_2\) are the same values found in Eq. (14) and the constants \(C\) and \(D\) are given by
\[
\begin{align*}
C &= \frac{\beta f(\beta - \omega^2)}{(1 + a)^2 \omega^2 + (\beta - \omega^2)^2}, \quad (23) \\
D &= \frac{-\beta f \omega \alpha}{(1 + a)^2 \omega^2 + (\beta - \omega^2)^2}. \quad (24)
\end{align*}
\]
In Eq. (22), \(A_1\) and \(B_1\) are arbitrary constants of integration which are to be fixed using the initial conditions. In order to keep the circuit operating in the negative conduction region (‘a’ represents the value of the negative slope) and to exhibit chaotic dynamics, as experienced from the experiment, the circuit parameters are chosen such that \((1 + a)^2 < 4\beta\). This makes the quantity \((1 + a)^2 - 4\beta\) within the square root in the expression for the eigenvalues in Eq. (14) always negative, so that \(\lambda_1\) and \(\lambda_2\) become complex and specifically, \(\lambda_2\) is the complex conjugate of \(\lambda_1\). Using the above values in Eq. (22), the solution that is a real valued function can be written in the standard form as
\[
y(t) = e^{-(1 + a)\frac{t}{2}} \left[ A_1 \cos \left( \frac{\sqrt{4\beta - (1 + a)^2}}{2} \right) t + B_1 \sin \left( \frac{\sqrt{4\beta - (1 + a)^2}}{2} \right) t \right] + C \sin \omega t + D \cos \omega t. \quad (25)
\]
FIG. 9: Analytically generated torus and period-doubling scenario at the frequency $\nu = 1.678$ kHz through analytical solutions of Eqs. (3) and (4) for fixed values of $a = -1.148$ and $b = 5.125$. The amplitude $f$ is the control parameter and the phase portraits are represented in the $(x-y)$ plane. (a) torus, $f = 0.3670$; (b) period-3TW, $f = 0.7881$; (b) chaos, $f = 0.7882$; (c) period-8T, $f = 0.7884$; (d) period-4T, $f = 0.7888$; (e) period-2T, $f = 0.8800$; and (f) period-1T, $f = 0.9400$. 
FIG. 10: Analytically obtained one parameter bifurcation diagram in the \((f - x)\) plane.

FIG. 11: Analytically obtained (i) phase portraits in the \((x - y)\) plane, and (ii) the corresponding power spectrum of \(x(t)\). (a) torus at \(f = 0.3658\), and (b) chaos at \(f = 0.7880\), through analytical solutions of Eqs. (3) and (4) for fixed values of \(a = -1.148\) and \(b = 5.125\).
Having found $y(t)$, $x(t)$ can be obtained using Eq. (20) as

$$x(t) = \frac{1}{\beta} \left[ e^{ut} \left( (vA_1 + uB_1) \sin vt + (vB_1 + uA_1) \cos vt \right) + \omega(C \cos \omega t + D \sin \omega t) \right],$$

where,

$$u = \frac{-(1 + a)}{2} \quad \text{and} \quad v = \frac{\sqrt{4\beta^2 - (1 + a)^2}}{2}.\quad (27)$$

The arbitrary constants $A_1$ and $B_1$ are now evaluated by using the initial conditions $(t, x, y) \to (t_0, x_0, y_0)$ in Eqs. (25) and (26). Upon solving, the values of $A_1$ and $B_1$ are obtained as

$$A_1 = \frac{e^{-ut_0}}{v} \left[ (y_0u - \beta x_0) + C(\omega \cos \omega t_0 - u \sin \omega t_0) - D(\omega \sin \omega t_0 + u \cos \omega t_0) \right] \sin vt_0 + y_0v - C\omega \sin \omega t_0 - Dv \cos \omega t_0 \cos vt_0, \quad (28)$$

$$B_1 = \frac{e^{-ut_0}}{v} \left[ (\beta x_0 - y_0u) + C(u \sin \omega t_0 - \omega \cos \omega t_0) - D(u \cos \omega t_0 + \omega \sin \omega t_0) \right] \cos vt_0 + y_0v - C\omega \cos \omega t_0 + Dv \sin \omega t_0 \sin vt_0. \quad (29)$$

(ii) **Regions** $D_+ (|x| \geq 1)$ and $D_- (|x| \leq -1)$:

In these regions, where we have positive slopes, the piecewise linear function $g(x)$ takes the value $g(x) = bx \pm (a - b)$. Then Eq. (3) and (4) become

$$\frac{dx}{dt} = f \sin(\omega t) - x - y - bx \pm (a - b), \quad (30)$$

$$\frac{dy}{dt} = \beta x. \quad (31)$$

Differentiating Eq. (31) once with respect to time, and using Eq. (30) in the resultant equation, we get

$$\frac{d^2y}{dt^2} + (1 + b) \frac{dy}{dt} + \beta y = \beta f \sin(\omega t) \pm \beta(b - a). \quad (32)$$

Equation (32) is also an inhomogeneous second order linear differential equation for which the solution can be obtained as in the previous region whose general solution is of the form

$$y(t) = A_2e^{\lambda_2 t} + B_2e^{\lambda_2 t} + C \sin \omega t + D \cos \omega t \pm \beta(b - a). \quad (33)$$
The values of $\lambda_3$ and $\lambda_4$ are found in Eq. (18) and $C$ and $D$ are given in Eqs. (23) and (24). In Eq. (33), $A_2$ and $B_2$ are arbitrary constants of integration which are to be fixed using the initial conditions. Since ‘b’ in $\lambda_3$ and $\lambda_4$ represents the value of the positive slope, the quantity $((1 + b^2 - 4\beta))$ remains positive. As a consequence $\lambda_3$ and $\lambda_4$ are real. Using the initial conditions, $(t, x, y) \rightarrow (t_0, x_0, y_0)$, the arbitrary constants $A_2$ and $B_2$ are determined as

$$A_2 = \frac{e^{-\lambda_3 t_0}}{(\lambda_3 - \lambda_4)} \left[ (\beta x_0 - \lambda_4 y_0 \pm \beta(b - a)\lambda_4) - (C\omega + D\lambda_4)\cos\omega t_0 + (D\omega + C\lambda_4)\sin\omega t_0 \right],$$

$$B_2 = \frac{e^{-\lambda_4 t_0}}{(\lambda_4 - \lambda_3)} \left[ (\beta x_0 - \lambda_3 y_0 \pm \beta(b - a)\lambda_3) - (C\omega - D\lambda_3)\cos\omega t_0 + (D\omega + C\lambda_3)\sin\omega t_0 \right].$$

Using $y(t)$, $x(t)$ can be written as

$$x(t) = \frac{1}{\beta} \left[ \lambda_3 A_2 e^{\lambda_3 t} + \lambda_4 B_2 e^{\lambda_4 t} + C\omega \cos \omega t - D\omega \sin \omega t \right].$$

If the initial conditions are chosen in the regions $D_\pm$, the system will end up in one of the fixed points $P^+$ or $P^-$ as the case may be, since they correspond to stable nodes.

The analytical solutions found in Eqs. (25) and (26) are plotted in Figs. 9 and 10. If we start with the initial condition $x(t = 0) = x_0$ and $y(t = 0) = y_0$ in the region $D_0$, the arbitrary constants $A_1$ and $B_1$ are evaluated at $t = 0$ using Eqs. (28) and (29). Then $x(t)$ evolves as given by Eq. (26) up to either $t = T_1$, when $x(T_1) = 1$ or $t = T'_1$ when $x(T'_1) = -1$. The value of $T_1$ and $T'_1$ are obtained numerically. Knowing whether $T_1 > T'_1$ or $T_1 < T'_1$, we can determine the next region of interest ($D_+$ or $D_-$). The arbitrary constants of the solutions of that region $A_2$ and $B_2$ can be evaluated using Eqs. (34) and (35) at time $t = T_1$ or in the time $t - T'_1$, at which the solution just enters into the region $D_+$ with $x(T_1)$ and $y(T_1)$ as initial conditions in Eqs. (33) and (36) (or the time at which the solution just enters into the region $D_-$, with $x(T'_1)$, $y(T'_1)$) and the solution evolves. This procedure can be continued for each successive crossing. The solutions given in Eqs. (25) and (26) for the inner region and Eqs. (33) and (36) for the two outer regions are matched for each successive crossing and phase portraits are represented in two different planes by slowly varying the normalized amplitude of the external signal ‘f’ (control parameter). We notice that when the external periodic signal is included ($f > 0$), initially the limit cycle appears and further when $f$ is increased gradually, the system exhibits torus breakdown to chaos. For example, $f = 0.3670$ torus is continuously observed up to $f = 0.7880$. Once the normalized amplitude of the external signal ‘f’ (control parameter) reaches the value of $f = 0.7882$, the torus breakdown and chaos appeared via a 3T window. On further increase
of the control parameter \( f \), reverse bifurcation appears, that is, for \( f = 0.7888 \), period 4T, for \( f = 0.8800 \), period 2T, for \( f = 0.9400 \), period 1T and finally reaches the boundary region where the dynamics stops. The above analytical results are summarized in phase portraits given in Fig. 9 in the \( (x - y) \) plane and a one parameter bifurcation diagram in the \( (f - x) \) plane in Fig. 10. The plot in Fig. 9 exactly matches with the experimental figures (See Fig. 6). The power spectrum corresponding to the analytically generated torus and chaos are given in Fig. 11. From this figure, the discrete set of frequencies corresponding to the torus and the broad band nature for the chaotic attractor have been observed and is naturally obvious. Figures 9 and 11 are the counterparts of those attractors obtained experimentally in Figs. 4(a), 4(b), 5(a), and 5(b).

Very recently, an interesting new test namely the ‘0-1 test’ to determine whether the dynamics of a deterministic system is regular or chaotic was proposed by Gottwald and Melbourne [29, 30], and it was used by several authors [20, 31, 32]. This test is universally applicable to any deterministic dynamical system, and it is applied directly to the time series data and hence does not require any phase space reconstruction. From the analytical solutions of Eqs. (2) for \( x(t) \) and \( y(t) \), one can write down the discrete time series data \( (t = n = 1, 2, 3, \ldots 1000) \). This discrete set of data is then used to calculate the translational variables \( p(n) \) and \( q(n) \), the mean square displacement \( M(n) \), and the asymptotic growth rates \( K_C \) and \( K \) using the formulae found in [32]. For the case of experimental analysis of the ‘0-1’ test, the voltage developed across the capacitor \( v_c \) is recorded using a 14-bit data acquisition system (Agilent U2531A 4ch 14 bits) at the sampling rate of 2MS/s. The experimental data is recorded for a chaotic attractor at amplitude \( F = 4.779 \) V and for a torus at \( F = 1.620 \) V, respectively. The recorded discrete data \( v_c(n) \) is used to calculate the translational variables \( p(n) \) and \( q(n) \), as defined in Ref. [32] as follows:

\[
p(n) = \sum_{j=1}^{n} v_c(j) \cos jc, \tag{37}
\]

\[
q(n) = \sum_{j=1}^{n} v_c(j) \sin jc,
\]

where \( c \) is a constant randomly chosen between \((0 - 2\pi)\), and \( n = 1000 \) is chosen for the analysis. For both our analytical and experimental data, we applied this ‘0-1 test’ and the results are presented in Figs. 11 and 12. From the Figs. 11a(i) and 11a(ii), it is observed that the trajectories in the \( (p, q) \) plane exhibit Brownian-like motion, implying that the dynamics is unbounded or chaotic (at \( f = 0.7882 \) with \( K = 0.956 \) for analytical data and \( F = 4.779 \) V with \( K = 0.997 \) for experimental data). The mean square displacement \( M(n) \) grows linearly in time both for analytical and experimental data, as shown in Figs. 11b(i) and 11b(ii), and indicates the chaotic dynamics. The asymptotic growth rate \( K_c \) of the MSD against \( c \), for both analytical and experimental data are shown in Figs. 11c(i) and 11c(ii), respectively. It fluctuates around one, implying the unbounded nature. The plot of \( K \) as a function of \( n \) is shown in Figs. 11d(i) and 11d(ii) for both analytical and experimental data. Naturally, we do not obtain \( K = 0 \) or 1 exactly, and the average value of \( K = 0.956 \) for analytical data and \( K = 0.997 \) for experimental data, indicating that the dynamics is chaotic. Similarly, it
FIG. 12: Chaotic or unbounded nature of dynamics for both analytical (Left panel (i)) and experimental (Right panel (ii)) data. (a) the behaviour of the trajectories is Brownian-like in the $(p, q)$ plane, (b) mean square displacement $M(n)$ grows linearly with time, (c) the asymptotic growth rate $K_c$ versus $c$ converges around one, and (d) plots of $K$ as a function of time $t$ fluctuates around one, indicating the unbounded nature.
FIG. 13: Torus or bounded nature of dynamics for both analytical (Left panel (i)) and experimental (Right panel (ii)) data. (a) the behaviour of the trajectories is bounded in the \((p, q)\) plane, (b) mean square displacement \(M(n)\) is bounded with time, (c) the asymptotic growth rate \(K_c\) versus \(c\) converges around zero, and (d) plots of \(K\) as a function of time \(t\) fluctuates around zero, indicating the bounded nature.

is observed from the Figs. 12a(i) and 12a(ii) that the trajectories in the \((p, q)\) plane exhibit regular dynamics, implying a torus at \(f = 0.367\) with an average value of \(K = 0.021\) for analytical data and at \(F = 1.620\) V with an average value of \(K = 0.108\) experimental data.
The Figs. 12b(i) and 12b(ii) show that the mean square displacement $M(n)$ is bounded in time for both analytical and experimental data, indicating that the underlying dynamics is torus-like. The asymptotic growth rate $K_c$ of the $MSD$ against $c$, for both analytical and experimental data are shown in Figs. 12c(i) and 12c(ii), respectively. It fluctuates around zero implying the bounded nature. A plot of $K$ as a function of $n$ is shown in Figs. 12d(i) and 12d(ii) for both analytical and experimental data, again it converges to zero implying the bounded nature.

V. CONCLUSION

In this paper, we presented analytical and experimental results of the dynamics of a forced parallel $LCR$ circuit with a simple nonlinear element. The circuit constructed with the help of this simplified nonlinear element having single negative conductance is studied experimentally, using the amplitude of the external periodic signal as the control parameter, for the chosen fixed circuit parameters. The results are captured on the oscilloscope screen and reported. Oscillographs exhibiting torus breakdown to chaos, reverse period doubling, intermittency, Poincaré maps, and their corresponding power spectra are shown in the respective figures. Further, the experimental results are confirmed by finding the analytical solutions through fixed point analysis. The figures corresponding to the analytical results match exactly with the figures drawn out of experimental observations. In conclusion, the hardware laboratory experimental observations and analytical results are in very good agreement with each other, also with a ’0-1 test’ for distinguishing chaotic and regular dynamics. Apart from our previous work [23], with this simple nonlinear element, we have demonstrated the torus breakdown to chaos and once again we proved that single negative dissipation is sufficient to generate chaotic dynamics in any nonautonomous circuit, both analytically and experimentally. Being a forced parallel $LCR$ circuit, it exhibits a rich variety of dynamics compared to the forced series $LCR$ circuit with the simple nonlinear element, and therefore the present study assumes much importance from the nonlinear dynamics point of view. Finally, the present study leads to the path to understand the possibilities of achieving novel networking systems, communication systems with adaptability to incorporate secure communication using chaotic signals, spatio temporal patterns in networks, torus synchronization, chaos on a torus and so on.

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References